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DIFFERENTIAL CALCULUS
FOR BEGINNERS

BY

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WITH AN INTRODUCTION

BY

SIR OLIVER LODGE, D.Sc., F.R.S., LL.D.

PRINCIPAL OF THE UNIVERSITY OF BIRMINGHAM

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PREFACE.

THE object of this book is to provide an easy introduction to the Calculus for those students who have to use it in their practical work, to make them familiar with its ideas and methods within a limited range.

A good working knowledge of elementary Algebra and Trigonometry is assumed, and in the latter part of the book it is supposed that the student will have acquired some knowledge of the methods of coordinate geometry.

Stress is laid at the outset on the graphic representation of functions, as by this means a vividness and reality is given to the processes of the Calculus which is not easily obtained without such ocular assistance. In this way the theory of maxima and minima becomes very simple and convincing.

It is hoped that the chapter on the graphic solution of equations will be found helpful and interesting.

The idea of integration (as the reverse of differentiation) is introduced into the chapter on successive differentiation.

Partial fractions have been treated somewhat fully, as the author feels that the acquirement of facility in the various methods of obtaining such fractions is of great educational value, besides being highly interesting.

The chapter on curvature is short, but will, it is believed, be found sufficient for ordinary practical purposes.

Indeterminate forms have been treated more fully than usual, and have been made to lead up to what appears to be the most powerful method of finding asymptotes to plane curves. In the chapter on tracing curves given by cartesian equations this method has been used for finding both linear and curvilinear asymptotes. Polar coordinates are dealt with in the last chapter, but some acquaintance with them is assumed in the latter part of the chapter on partial differentiation.

I have adopted the method of differentials at the commencement, and made it the basis of the whole work, having found that this method is most useful and helpful to the student of Physics and Mechanics.

It is hoped that matters of primary importance in a first book on the Calculus have been sufficiently dealt with, but it has been difficult in some cases to decide what ought or ought not to be omitted.

Numerous illustrative examples will be found throughout the book, some of which are of intrinsic importance, while some are intended to prepare the student for further developments.

I have derived much help from Professor Perry's *Calculus for Engineers*, particularly in dealing with exponential functions. I desire to express my thanks to my brother, Sir Oliver Lodge, for his valuable introduction, which indicates the great range of subjects that are open to the student who has a fair working acquaintance with the Calculus. I cannot express too strongly my deep obligations to Mr. P. J. Heawood, Lecturer in Mathematics in the University of Durham, who has kindly revised all the proof sheets with scrupulous care, and has given many valuable criticisms and suggestions.

COOPERS HILL, *June*, 1902.

PREFACE TO THE SECOND EDITION.

I have to acknowledge with thanks the receipt of several valuable suggestions and corrections, most of which are incorporated in the present edition.

CHARTERHOUSE, *April*, 1905.

PREFACE TO THE THIRD EDITION.

In this Edition, a set of Miscellaneous Examples have been added, for many of which I am greatly indebted to the Rev. J. F. Twisden, Mr. C. S. Jackson, and other kind friends. Some of the examples, marked [C.S.C.], have been taken from papers set by the Civil Service Commissioners, and are reprinted by permission of the Controller of H.M. Stationery Office. It is hoped that these examples will be useful for purposes of revision, both for army candidates, and for scholarship students.

CHARTERHOUSE, *June*, 1908.

PREFACE TO THE FOURTH EDITION.

The chief emendations are in connection with a^x, and Taylor's Theorem. Also a few examples of special interest have been added to some of the sets.

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INTRODUCTION

BY SIR OLIVER LODGE, F.R.S.

THE branch of Mathematics called the Differential Calculus was originally invented by Sir Isaac Newton, and later independently by Leibnitz, for the purpose of dealing with variable quantities; that is to say, with any quantity which does not remain constant, of which kind of quantity the whole of Physics and Engineering and daily life is full. For in so far as things remain constant they are stagnant; every kind of activity involves variable quantities, and therefore to every kind of activity the differential calculus is applicable. It is called the differential calculus because it attends not so much to the quantities themselves as to their variations, their increases or differentials: it is the calculus which deals with differentials, especially with the ratio of two differentials to each other. For by a differential is understood an infinitesimal difference or increase or increment (or, for the matter of that, decrease or decrement), and whenever one quantity changes there is always some other quantity which changes too, the two quantities being called connected variables; or either of them is called a function of the other.

Although it is customary to attend to very minute increments, and to consider each differential as infinitesimal, yet the ratio of two such microscopic differentials will in general be finite, and may be large. Thus, if a thing advances $\frac{1}{100}$ part of an inch in the millionth part of a second, it is then going at the rate of 10,000 inches per second, much faster than an express train, and nearly as fast as a bullet. Or if a slope descends the thousandth of a millimetre for each hundredth of a millimetre along, it has a gradient of 1 in 10, and is too steep for a railway without cogs.

Or, if a rod expands the millionth part of its length for one-tenth of a degree rise in temperature, it has about the expansibility of iron. In this last example the two connected variables are the temperature and the length of the rod.

It may be asked, why deal with infinitesimal quantities at all? Why not attend to appreciable changes of magnitude and take their ratio? If we could depend on quantities varying uniformly, or if they always bore to one another the relation of simple proportion, this would be the natural and sufficient thing to do. But in practice it is only a few quantities which are thus simply connected, and if we were constrained to attend always to finite differences their ratio would in general give us a mere average result, not an actual result at any instant.

To know that a bullet has travelled a mile in ten seconds does not tell us with what speed it left the muzzle; and instruments adapted to ascertain this or any other actual velocity must be chronographic instruments able to record extremely small increments of time and the corresponding moderately small distance travelled.

In the laboratory it is to be observed that we are bound to deal with finite changes, and thus are limited to a kind of average result: we may make the observed intervals small, but we cannot make them infinitesimal. But in theory we are not so limited, and the theoretical treatment of infinitesimal changes is decidedly simpler and easier than the treatment of finite changes; except when the observed quantities are varying at a steady or a proportional rate. In that case the finite difference becomes as easy to deal with as the differential.

It will be observed that these differentials or infinitesimal differences, though simple in theory to deal with, and though they are the quantities primarily considered in the calculus, are not individually important except indeed in connection with approximate calculations. It is the ratio of one differential to another that is for many purposes the important thing: the ratio of the distance travelled to the time taken, or the ratio of ascent to horizontal distance as we climb a hill, or the ratio of the expansion of a body to the rise of temperature which causes or accompanies that expansion; in such cases it is the comparison or relation between the changes of the connected variables, to which attention is directed, rather than the changes themselves; and in the limit when the changes are infinitesimal or evanescent, the above ratios denote, respectively, the speed of travel at a given instant, the slope or gradient of the hill at a given point, and the expansibility of the body at a given temperature.

Newton called the rate of variation of a flowing or variable quantity its "fluxion"; now-a-days it is commonly called its "differential coefficient," which is a long and inconvenient term, but is intended to be and to sound more general and inclusive than any of its special cases, such as "speed," or "slope," or "expansibility": differential coefficient is in fact the most general version of these terms, and is denoted by some such symbol as $\frac{dy}{dx}$ or $\frac{dy}{dt}$, where dy, dx, and dt denote infinitely small changes of any connected variables y and x, or y and t.

One of the objects of the present book is to explain the methods by which the ratio between the differentials of connected variables can be obtained, whatever may be the mathematical equation by which they are connected with each other, and to give some illustrations of the use of such ratio when obtained.

Integral Calculus.

Often it is not the rate of change of a quantity that has to be attended to, but the change itself; not an infinitely small change, but a change of some magnitude. Such finite change is built up of a succession of differential changes. Although each differential is infinitesimal, the sum of a sufficient number of them will become finite.

Thus the total expansion of a body for the rise of a good many degrees may be obviously an important quantity; the total height of a hill may be considerable; and so also may the actual distance travelled in a finite time. The connection between these quantities and the rates at which they grow is an important thing to investigate. These whole quantities are called integrals, and the mode of obtaining them from the given slope or expansibility or speed, by adding the differentials from moment to moment, or from degree to degree, or from step to step, is called integration; and the investigation of the methods by which integration is performed in various cases is the subject of the Integral Calculus.

Integration is therefore a kind of addition, and the idea of it is even simpler than the idea of the ratio involved in slopes or expansibilities or speeds.

But the actual *process* of integration is not found in practice to be so simple. It is much easier to differentiate than it is to integrate, just as it is rather easier to multiply than to divide, or to raise to a power than to extract a root. The work of integration is not done by actual addition, but by a kind of backward process, which here can be best explained by an example.

Thus: If we are given an expression for the height of a falling body at any instant, we should be able, by means of the differential calculus, to calculate an expression for its speed at any instant. In fact, the expression for the speed is the differential coefficient of the expression for the height.

If, on the other hand, we are given the expression for the speed, we should be able, by means of the integral calculus, to obtain the expression for the height at any time. We should, in fact, have to find or recognize the expression whose differential coefficient is given. It is, therefore, a backward process. Some methods of performing this process will be given in this book, but for more general methods the student must wait till he can read a book on the integral calculus.

Another way of finding an expression for the finite increase of a variable function, dependent also on the expression for an infinitesimal increase, will be given in this book in the chapter on Taylor's and Maclaurin's series. This method is strictly a method belonging to the differential calculus, though it has interesting applications in connection with approximate integration.

Other matters connected with the differential calculus are those connected with the plotting of curves, finding of tangents, normals, asymptotes, curvature, and such like; an important subject, as it is very necessary to be able to plot a curve, and to realize its properties, when its equation is given: more particularly since the relations between any two connected variables can be very conveniently depicted by means of a curve drawn so that the coordinates of any point on the curve represent a pair of values of the variables. Many examples will be found in the following chapters. This graphic method of representing functions is largely used by Physicists and Engineers.

Differential Equations.

To illustrate further the advantage of being able to treat subjects familiarly by the differential calculus, and to employ operations which are compactly symbolized by its notation, we may proceed to problems involving a little more knowledge of mechanics.

Taking the symbol v to represent velocity, it is manifest that $\frac{dv}{dt}$ must be acceleration, and $m\frac{dv}{dt}$ will therefore be force. The force involved in falling stones and such like is *constant*, viz. their weight. The simplest examples of *variable* forces are afforded by plucked wires or struck bells or slightly deflected pendulums. In each of these cases there is a restoring force depending on the displacement, and if the displacement is called x, the velocity (v) will be $\frac{dx}{dt}$, and the force will be $m\frac{dv}{dt}$, or, as it may also be written,

$$m\frac{d\left(\frac{dx}{dt}\right)}{dt} \text{ or } m\frac{d^2x}{dt^2}.$$

In the simplest, and all ordinary cases, this force is proportional to the displacement, a law observed in elastic bodies by Hooke.

We cannot say that the force is *equal* to the displacement: that would manifestly be absurd, because they are not quantities of the same kind; but if the displacement is multiplied by an appropriate factor, not a mere numerical factor, but a factor which involves in the elastic case the material and dimensions of the spring, or in the stretched string case its length and tension, or in the pendulum case the intensity of gravity and the length of the pendulum; and if this appropriate factor be called k, then it will be legitimate to say that the restoring force is equal to k times the displacement, that is to say, $F=kx$; or rather, in order to express the fact that it is a restoring force urging the body back to its original position, and not a sort of repulsive force driving it further and further away, we shall write $F=-kx$. The force, in fact, always acts in a direction opposite to the displacement, and hence must be given the negative sign.

So the facts of the case are expressed by the equation $\frac{mdv}{dt}=-kx$, or, to avoid the apparent introduction of three variables, of which v is only an intermediate and manifestly connected one $\left(v=\frac{dx}{dt}\right)$, the equation is usually written

$$m\frac{d^2x}{dt^2}+kx=0.\qquad\qquad(1)$$

Now this is called a differential equation, and, in fact, a differential equation of the second order, because of the double differentiation involved in acceleration. But, nevertheless, it is a very simple equation: it is the equation of simple Harmonic Motion, the motion of any simply vibrating or sounding body.

One of the things we have to learn is how to pass from a differential equation to its integrated form, the form which deals explicitly with the quantities themselves, not with their changes.

This we do by the process of integration, not always by straightforward integration but generally by other operations based on the processes of the differential calculus; and the result in the above example is

$$x=a\cos(pt+\epsilon),\qquad\qquad(2)$$

where $p=\sqrt{(k/m)}$. This is the integral equation between x and t corresponding to the above given differential equation.

To the mathematician each of these two equations implies the other: they differ only in form; they express precisely the same facts; either may be deduced from the other, one such deduction being made by the process of differentiation, the other or inverse deduction being made by the process of integration.

In order further to call attention to the importance and interest of the problems which can thus be treated, I will mention two more examples.

The above simple harmonic motion, which might be plotted as a sinuous line with amplitude a, is of the kind known as perpetual. There is no friction or other term to dissipate energy. If such a term be introduced into the differential equation, say a frictional force proportional to the speed, then the amplitude becomes a function of the time, for instance ae^{-rt}, and this would represent a damped vibration, and be plotted as a sinuous curve gradually dying out with logarithmically decaying amplitude, *i.e.* dying out as a decreasing geometrical progression. Such oscillations occur in electricity, as well as in tuning-forks, and their importance has been emphasized by the discoveries of Hertz.

Another illustration of differential equations may be given here in order to emphasize its importance, viz. the case when there are two independent variables, and when you are studying fluctuations of a third variable dependent on these two.

For instance, when we are discussing the height of an object above some fixed level: it may vary from place to place, as the ground does in undulating country, or it may vary from time to time like the height of a balloon or cloud, or like the height of the ground if it were subject to earthquake disturbances. And these two modes of variation, viz. time and horizontal distance respectively, are quite disconnected and independent. But if we look at the surface of the sea we shall observe the effect of the two in combination. Imagine the liquid frozen at any instant and you could travel about it as on undulating country. Shut your eyes and let your boat rest on the surface and you are conscious merely of an oscillation up and down. The rate at which you oscillate up and down depends on circumstances beyond your control; the rate at which you row is an independent circumstance which you can determine for yourself. If you attend to the motion of the water itself, it appears to travel horizontally, though in reality each particle merely swings up and down or round and round a vertical circle. But the wave-*forms* do travel horizontally, and their rate of travel is somehow connected with the rate of swing of the particles up and down. To find out the whole of the circumstances of this complex or wave motion we must analyze its causes and write down a corresponding differential equation and then integrate it.

The necessary differential equation is one which expresses the fact that the force urging the water to move may be regarded from two points of view—the active or elastic or gravitational restoring-force mode, which regards the force as proportional to the curvature of the surface, say $k\frac{d^2y}{dx^2}$ (for small waves), and the passive or inertia mode of regarding the reaction to that force, as mass-acceleration, $m\frac{d^2y}{dt^2}$, where y is the height of the element of water surface under consideration.

Writing these two expressions as action and reaction equal to one another, we have

$$k\frac{d^2y}{dx^2}=m\frac{d^2y}{dt^2};$$

where on the left hand side y is regarded as changing only with the variable x, as if the surface were frozen, or rather as if an instantaneous photograph were taken of it; and where on the right-hand side we attend only to those changes of y which are due to efflux of time alone, all horizontal motion being ignored, and the particles which come into vertical line from instant to instant alone being contemplated.

Such equations, where two causes of variation of a single quantity y are independently considered, are called partial differential equations, and their treatment is exceedingly important though rather difficult. The result of integration in this case is to give the solution in the form $y=f(x\pm Vt)$, where V is the velocity of propagation of the waves, and is equal to the square root of the ratio of k to m.

It is not to be supposed that these examples will be followed by any one new to the subject; and, moreover, they contain mechanical difficulties as well as difficulties of calculus notation and methods; but they are taken as typical examples to illustrate the kind of power which will be conferred upon the student by acquaintance with the differential or infinitesimal calculus, and to indicate to the student of elementary physics some of the important varieties of physical problems which can by its means be readily and familiarly dealt with.

CHAPTER I.

FIRST PRINCIPLES—GRAPHIC REPRESENTATION OF FUNCTIONS.

THE differential calculus treats primarily of the relative increments of mutually related quantities when such increments are infinitely small. Thus, if a square sheet of metal or a circular metallic plate is slightly expanded by heat, the question as to the mathematical relation between the increase of area and the increase of the side of the square or of the radius of the circle is a question belonging to the differential calculus.

These small increments are called '*differentials.*' The differential of any quantity x is written for brevity $d(x)$ or dx, the letter d being an abbreviation of the expression *differential of*, which means *infinitely small increment of.*

Example. Let x be the length of the side of a square, whose area will therefore be equal to x^2, and let the increase of area be required when the side receives an infinitely small increment.

If we do not use the notation of the calculus, we should say, let the increased length of side be x', then the increased area is x'^2; $\therefore$ the increase of area is $x'^2 - x^2$, and the increase of side is $x' - x$. Also, since $x'^2 - x^2 = (x' + x)(x' - x)$, and since, when x' and x differ by an infinitely small amount, $x' + x$ differs infinitely little from $2x$, we see that the increment of area is ultimately equal to $2x(x' - x)$.

If we do use the notation of the calculus, we should denote the increased length of side by $x + dx$, and therefore the increased area would be $(x+dx)^2$ which is $x^2+2x.dx+(dx)^2$; $\therefore$ the increase of area is $2x.dx+(dx)^2$. If we now bring in the condition that dx is to be infinitely small, we see that $(dx)^2$ is infinitely smaller than $2x.dx$, since it is the product of *two* infinitely small factors. Therefore, finally, we have the increase of area $=2x.dx$, which is the same as $2x(x'-x)$ obtained previously.

The essential difference between the two methods is that in the first method we give a name (x') to the increased length of side, whereas in the calculus method we give a name (dx) to the increase *in* the length. This apparently trivial matter is of fundamental importance.

2. Meaning of the sign of equality. Two quantities are equal if their ratio is unity, and their difference is zero. In the case of finite quantities, if one condition is satisfied, the other is also. In the case of infinitely small quantities their difference is ultimately zero, but they are not to be considered equal unless their ratio is unity. Thus when we say that $2xdx+(dx)^2$ is ultimately equal to $2xdx$ we mean that their ratio is ultimately $=1$, which is evidently the case, since the surplus fraction $\frac{(dx)^2}{2xdx}$, which is equal to $\frac{dx}{2x}$, ultimately vanishes.

The rule (based on the above method of comparison) which will guide us in neglecting relatively small quantities is very simple, and is as follows: To obtain the ultimate value of an expression containing finite and infinitely small quantities, first perform all subtractions that may be possible, and then, if an infinitely small quantity is added to a finite one, the small quantity is relatively unimportant, and is to be omitted. Thus, in the above example, in reducing the increase of area, viz. $(x+dx)^2-x^2$, to its simplest form, the quantity x^2 is cancelled by subtraction, leaving the expression $2x.dx+(dx)^2$ which is equal to $(2x+dx)\,dx$, and this in the limit reduces to $2x.dx$, since $2x+dx$ is ultimately equal to $2x$.

Another way of considering the question is to remember that although dx is to be infinitely small, it is yet considered of sufficient importance to be taken into account, as obviously it is the supposed increase of side on which the whole problem depends, and therefore $2x\,dx$, which is a finite multiple of dx, is also important; but the *square* of dx, being a product of *two* infinitely small factors, is unimportant. In fact, $2x.dx$ and dx are of the same order of smallness, since their ratio is finite, but $(dx)^2$ and $2x.dx$ are not of the same order of smallness, since the ratio of the first to the second is ultimately zero. All these small quantities are included in the general name 'infinitesimals,' dx and $2xdx$ being called infinitesimals of the 1st order, and $(dx)^2$ being called an infinitesimal of the 2nd order.

3. Orders of infinitesimals. The numerical magnitudes of small quantities can be compared only by means of their ratios. If their ratio is finite, they are said to be infinitesimals of the same order, otherwise they are infinitesimals of different orders.

Thus, if dx is an infinitesimal of the 1st order,
$(dx)^2$ .. .. 2nd ..
$(dx)^3$,, ,, 3rd ,,

and so on; and in any equation, if infinitesimals of different orders are added together, those which are of high order are unimportant compared with those of lower order; and, lastly, infinitesimals of any order, when added to finite quantities, are unimportant. When, therefore, unimportant terms are cut out, all the terms of an equation will be finite, or all the terms will be infinitesimals of the 1st order, or all will be of the 2nd order, and so on. We shall have no equations in which different orders will be mixed.

Thus in the equation $area = x^2$, the terms are finite. In the equation $d(\text{area}) = 2xdx$, both terms are infinitesimals of the 1st order.

4. We may show in a figure the retained, and the omitted, parts of the increment of the area of the square.

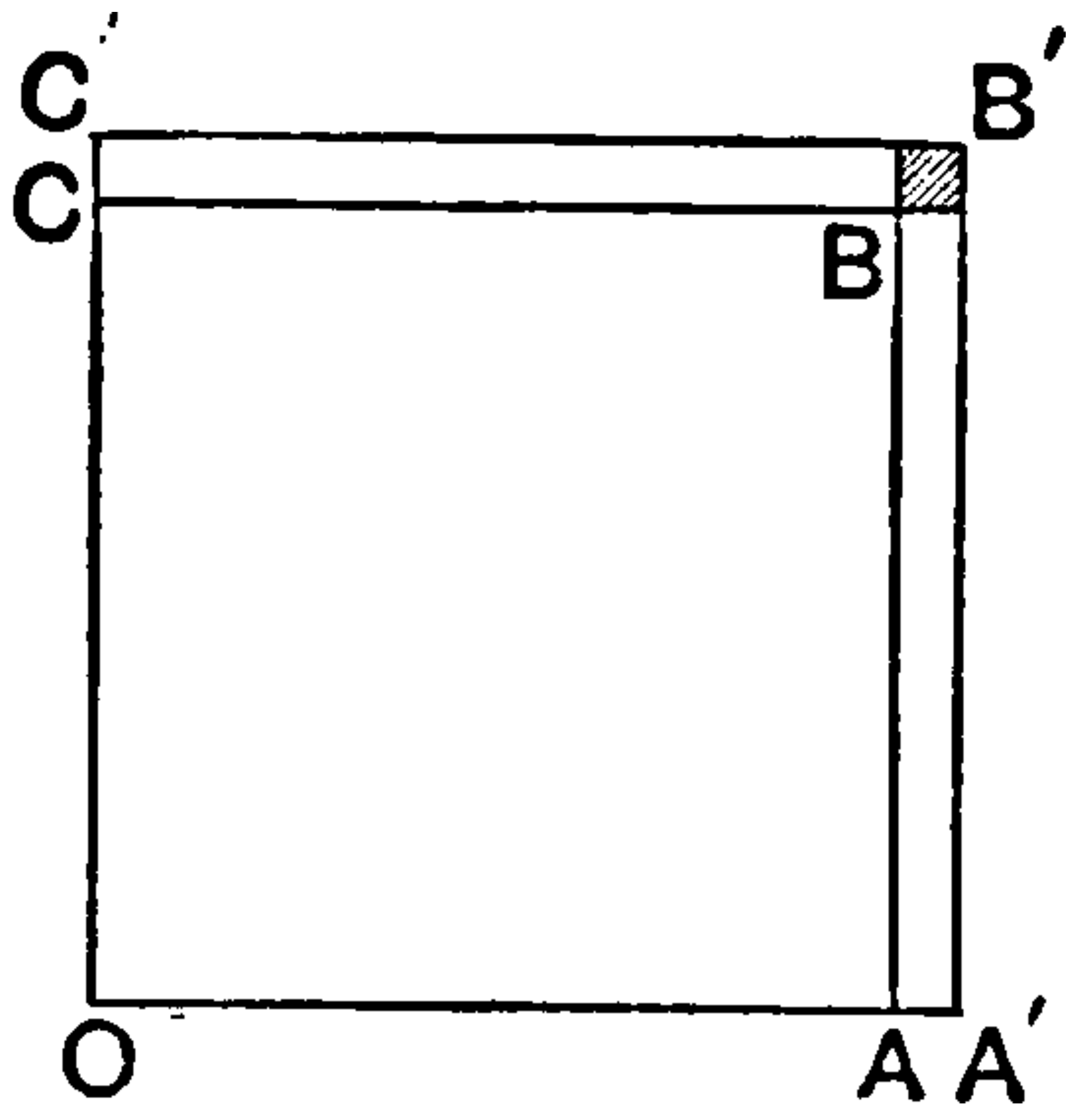
Thus, let OABC be the original square whose side $OA = x$.

And let O′A′B′C′ be the increased square (the increment being infinitely magnified in the figure to enable it to be seen).

Then $AA' = dx$,

and the areas of the rectangles BA′ and C′B are each $= x.dx$, so that their sum $= 2xdx$.

Lastly, the small square BB′ is $=(dx)^2$, and is ultimately infinitely smaller than either rectangle, since it will require an infinite number of such small squares to make up one of the rectangles.



Of course, the student must exercise his imagination, and think of AA′ as infinitely small, so that the rectangles BA′ and C′B are infinitely thin, thinner than the finest ink line that could be drawn, while the square BB′ is not only infinitely thin, but also infinitely short.

5. The calculus would record the above result briefly as follows:

If
$$y=x^2,$$
$$dy=2x\,dx,$$
or, more briefly still,
$$d(x^2)=2x.dx.$$

This result has wider application than merely to the area of a square. It is true, whatever x may be, that the differential of its square $=2x$ times the differential of x itself.

Thus, find the differential of the area of a circle whose radius, r, receives an infinitely small increment, dr.

The increase of area is evidently $\pi(r+dr)^2-\pi r^2$,

$$i.e.\ \ \pi\{(r+dr)^2-r^2\},$$

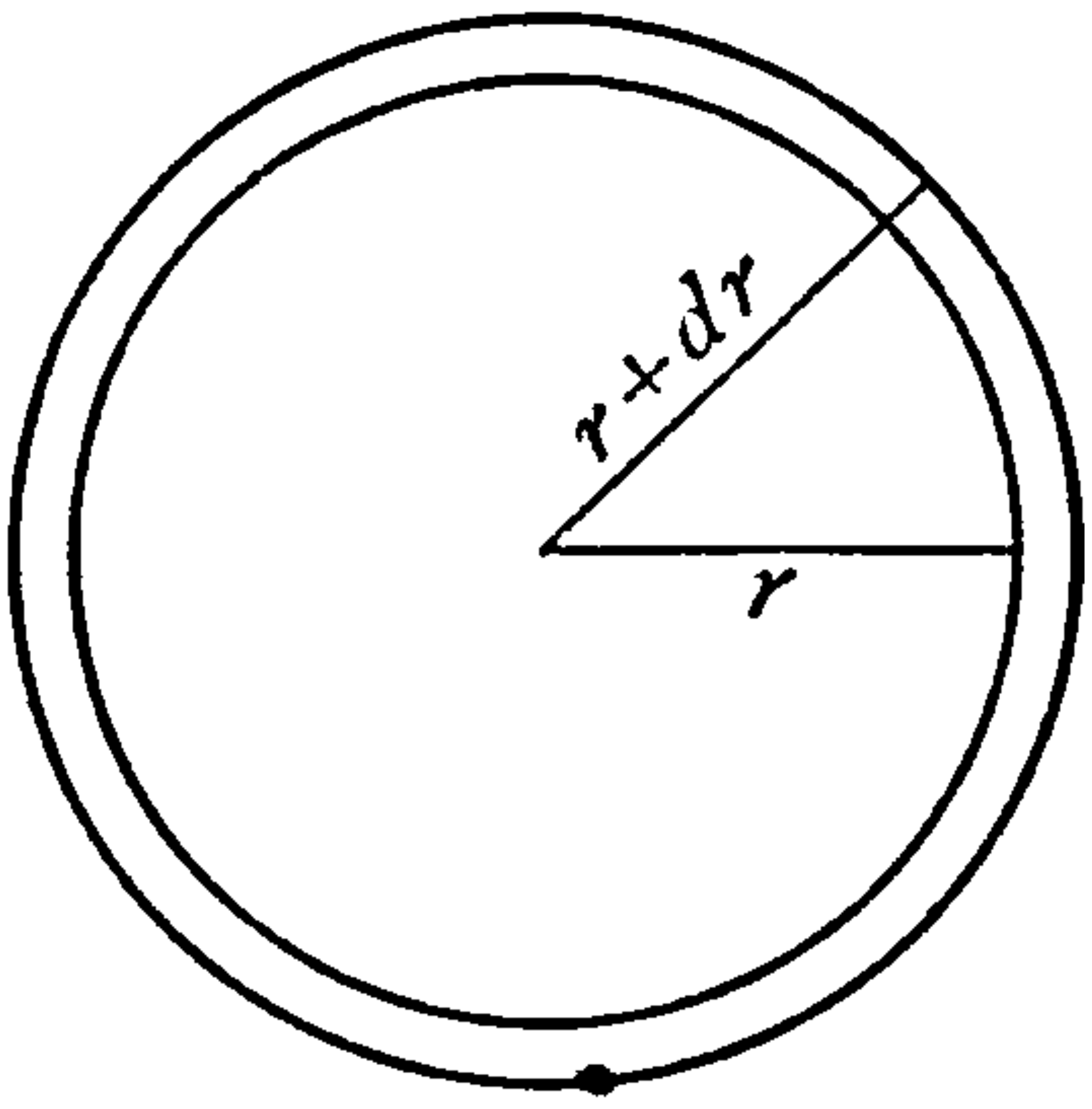
and the quantity in the brackets { } is $d(r^2)$ which $=2r.dr$ just as $d(x^2)=2x.dx$;

$$\therefore \text{ the increase of area}=2\pi r\,dr.$$

The steps of the work may be stated as follows:

$$d(\pi r^2)=\pi d(r^2)=\pi 2r\,dr=2\pi r\,dr.$$

If we look at the same problem geometrically we shall see the reasonableness of this result, for $2\pi r$ is the circumference of the circle, and dr the width of the added strip, therefore the area of the added strip $= 2\pi r.dr$, if we remember that dr



is infinitely small. Of course for any finite increase in r there would be a small error in this result, but the error is an infinitesimal of the second order, and is therefore of less and less relative importance the smaller we take dr.

Example. Find the difference between πr^2 and $\pi(r+dr)^2$ when $r=4$ inches and (1) $dr = \cdot 1$ inch; (2) $dr = \cdot 01$ inch; (3) $dr = \cdot 001$ inch. Compare the result in each case with the corresponding value of $2\pi r\,dr$.

Ans. The ratios of the actual increments to those given by $2\pi r dr$ in the several cases are

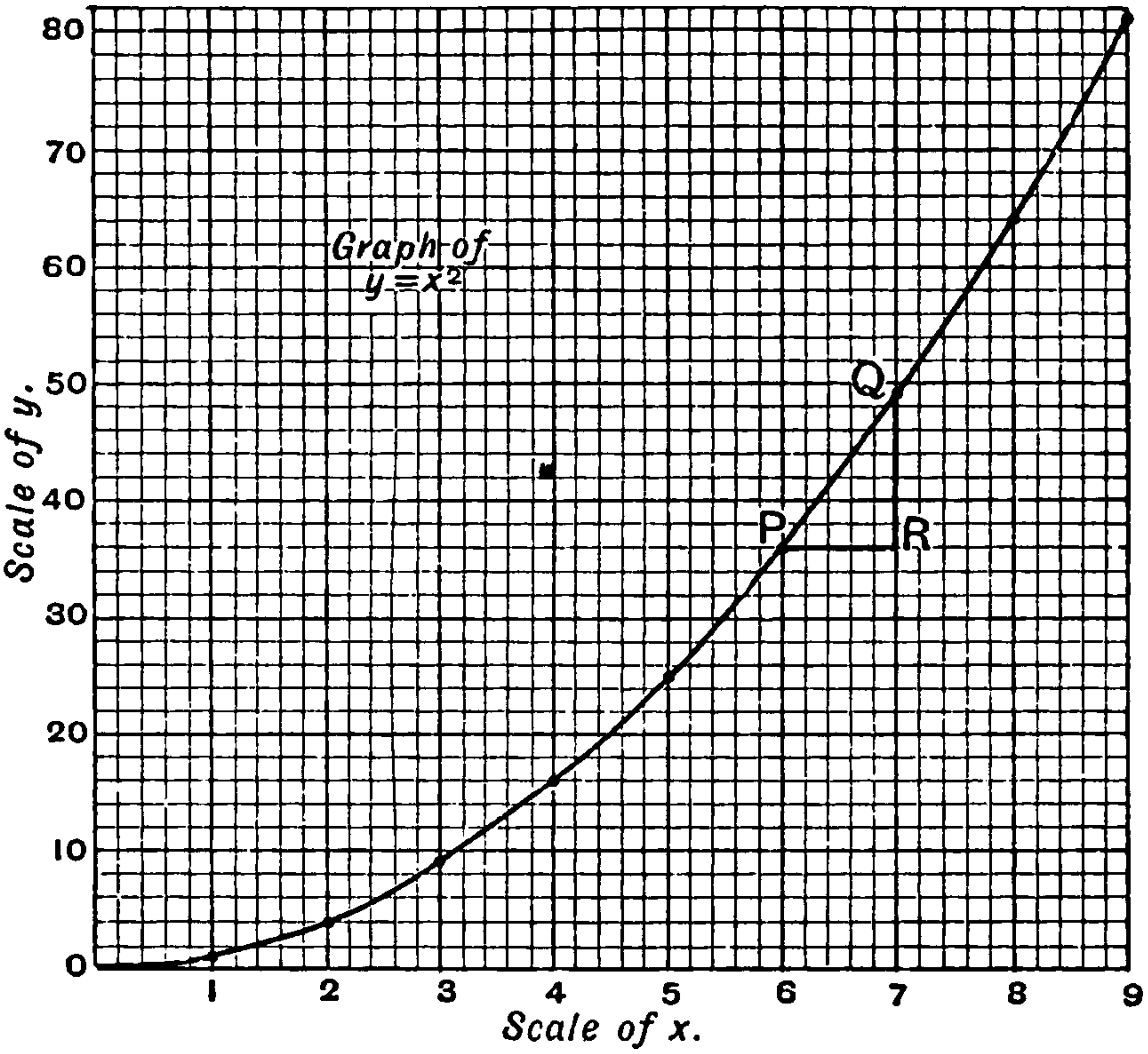
(1) $1\frac{1}{80}$; (2) $1\frac{1}{800}$; (3) $1\frac{1}{8000}$.

6. Graphic mode of representing the variations of two mutually connected variables.

If two variables are connected, as, for example, the side and the area of a square, their comparative numerical values can be represented artificially by means of a curve, in the following way. Let x and y be the two variables, each measured in terms of convenient units, *e.g.* in the case of the square, we should have $y = x^2$, when y is measured, say, in square inches, and x in inches. Then if we measure off along a horizontal line, from some fixed point O, lengths representing the different values of x, and erect perpendiculars to this line, whose lengths shall represent the different values of y, we can draw a curve connecting the ends of these perpendiculars. This curve is called the graphic representation (or shortly, the graph) of y considered as a function of x. We will illustrate by the above example, $y = x^2$.

The following table gives a series of corresponding values of x and y, and these are plotted in the adjoining diagram. The height of the curve at intermediate points gives values of y corresponding to intermediate values of x.

x	y
0	0
1	1
2	4
3	9
4	16
5	25
6	36
7	49
8	64
9	81



There is no need to take the same scales for x and for y; each should be taken on a scale which is most convenient, having regard to the numerical range of the quantity requiring to be shown. The curve representing the function is a parabola.

7. **Gradient of the** curve $y=x^2$.

The student should notice that the steepness of the curve $y=x^2$ increases as the numbers get larger. This is due to the fact that, when x is large, a given increase in x causes a larger proportional increase in its square. Thus, if x increases from 2 to 3, the square increases from 4 to 9, which is an increment of 5, whereas if x increases from 8 to 9, the increment of its square $=81-64=17$, which is much greater, and if x changes from 20 to 21, the increase of the square is much greater still. For any two values of x the average rate of change of x^2, compared with the change of x, is obtained by dividing the change in x^2 by the change in x, and will be shown in the diagram by the tangent of the angle of slope of the chord joining the two points. The tangent* of this angle is called the 'gradient' of the chord, a convenient term introduced to avoid the long equivalent phrase, 'the tangent of the angle which the chord makes with the x-axis.' Similarly the gradient of the curve itself at any point is defined as the tangent of the angle which the tangent to the curve at that point makes with the x-axis. It is the limiting case of the gradient of a chord joining two points P and Q when Q moves close up to P.

In the above example, if x changes from 6 to 7, the gradient of the chord is 13 vertical units divided by 1 horizontal unit, since $7^2 - 6^2 = 13$, and $7 - 6 = 1$. We shall call this a gradient of 13, though the tan of the angle depicted in the diagram will not be actually 13 unless the horizontal and vertical scales are the same. All we have to remember is that in any fraction giving the gradient, the numerator is measured in vertical units, and the denominator in horizontal units. Thus a gradient of 13, *i.e.* $13 \div 1$ is, as already stated, 13 vertical units to 1 horizontal unit. If this is borne in mind, the difference in the two scales will present no difficulty. The gradient 13 is represented by the ratio QR : PR in the figure.

To find the gradient of the *curve* at any point P, we must find the gradient of the chord PQ when Q is infinitely close to P. Now, we have proved that when x receives an infinitely small increment dx, the change in x^2 is $2x.dx$, therefore the required gradient is $2x.dx \div dx = 2x \div 1$. For example, when $x = 6$, the gradient of the curve is 12, and when $x = 7$, the gradient is 14. Hence, in this case, while the gradient of the chord joining the two points in the figure is 13, the gradient of the curve itself gradually changes from 12 to 14.

8. It will be well to see what is the physical meaning of the gradient of the curve at any point. The equation $y = x^2$ may be looked upon as a merely numerical relation between a number x and its square. This is how we have just been considering it. But we may also take it as the relation between the side and the area of a square plate. In that case y is so many square inches and x is so many inches. The gradient of the curve is given by the equation:

---

* The numerical value of the tangent of the angle is to be taken as it would be if the vertical and horizontal units of scale were the same length. This numerical value is always meant when the gradient is spoken of. The distorted angle due to unequal scales is a mere accident of the diagram, and is unimportant.

$$\text{gradient} = \frac{dy}{dx} = \frac{2x}{1},$$

and, if we imagine the area of the square growing by a gradual increase in the length of its variable side x, the value of the gradient for each value of x represents the rate of growth of the area *per* unit increase in the side, *i.e.* so many square inches of area *per* inch growth of side.

For example, when $x = 6$ inches, the gradient is 12 square inches *per* inch, and when $x = 7$ inches, the gradient is 14 square inches *per* inch. These are the actual rates at which the square is growing at the two instants when its dimensions are those which correspond to each end of the chord PQ in the figure, and the *average* rate of growth between these two points is the gradient of the chord, which is 13 square inches per inch.

Hence, the gradient of the curve represents the rate of growth of y compared with that of x, and is described as so many vertical units *per* horizontal unit.

We have arrived at this notion of a gradient by considering the special curve $y = x^2$, but the reasoning is perfectly general, and we will therefore state it again below in connection with any two related variables, x and y. But first we will introduce a term of much use in abbreviating the statement that there is a relation between the variables.

9. Function. Explicit and implicit functions. If two variables x and y are so related that a change in one of them necessitates a change in the other, they are said to be *functions* of each other. Thus the area of a square and the length of its side are functions of each other, and so are the volume of a sphere and its radius. If one of the two variables is directly expressed in terms of the other, it is usual to say that the one variable is a function, or an explicit function, of the other: *e.g.* if $y = x^2$ we say that y is an explicit function of x. We may of course express x as an explicit function of y, viz. $x=\sqrt{y}$.

If y is expressed as a function of x, y is called the subordinate, or the dependent, variable, and x is called the principal, or the independent, variable. The latter term is not a happy one, as of course it is not independent of y, but so long as the student realizes this, the term does very well as an antithesis to the term 'dependent.'

If the variables occur mixed together in an equation, as for example in any of the following cases,

$$2x+3y=6,$$
$$x^2+y^2=4,$$
$$xy=c^2,$$

each is said to be an *implicit* function of the other. We may express either of them as an explicit function of the other if we desire it. In some cases, however, such transformation of the equation may be very difficult, and may also be undesirable.

EXAMPLES.

1. If $y^2-2ay+x^2=0$, make y an explicit function of x.

Ans. $y=a\pm\sqrt{(a^2-x^2)}$.

2. If $1+\log_a y=2\log_a(x+a)$, make y an explicit function of x.

Ans. $y=(x+a)^2\div a$.

3. Given $\tan^{-1}x+\tan^{-1}y=\alpha$, make y an explicit function of x.

Ans. $y=\dfrac{\tan\alpha-x}{1+x\tan\alpha}$.

4. Given $y=\dfrac{2x-1}{3x-2}$, make x an explicit function of y.

Ans. $x=\dfrac{2y-1}{3y-2}$.

10. If we wish to denote that y is some function of x, without specifying what function, it is usual to make use of the first letter of the word *function*, and to write $y=f(x)$. Or we may use other similar letters, and write $F(x)$, or $\phi(x)$, to denote functions of x. It is merely a convenient notation, with which the student will soon become familiar. The value of the function $f(x)$ for particular values of x, such as 1, 2, 3, etc., will be denoted by $f(1)$, $f(2)$, $f(3)$, etc.

Thus if $\quad f(x)=x^2$

we shall have

$$f(1)=1$$
$$f(2)=4$$
$$f(3)=9$$

and so on.

The student should work the following examples, so as to become familiar with the notation.

EXAMPLES.

1. If $f(x)=x^2-3x+2$, find $f(1)$, $f(3)$, $f(0)$, and $f(a)$.
Ans. $0, 2, 2, a^2-3a+2$.

2. If $f(\theta)=\cos\theta$, write down the values of $f(0), f\left(\frac{\pi}{3}\right), f\left(\frac{\pi}{2}\right), f(\pi)$.
Ans. $1, \frac{1}{2}, 0, -1$.

3. If $F(x)=2^x$, find $F(2)$, $F(3)$, $F(0)$, and $F(F(2))$.
Ans. 4, 8, 1, 16.

4. In the first example write down $f(x^2)$, $f(\surd x)$, $f(x+h)$.
Ans. x^4-3x^2+2, $x-3\surd x+2$, $(x+h)^2-3(x+h)+2$.

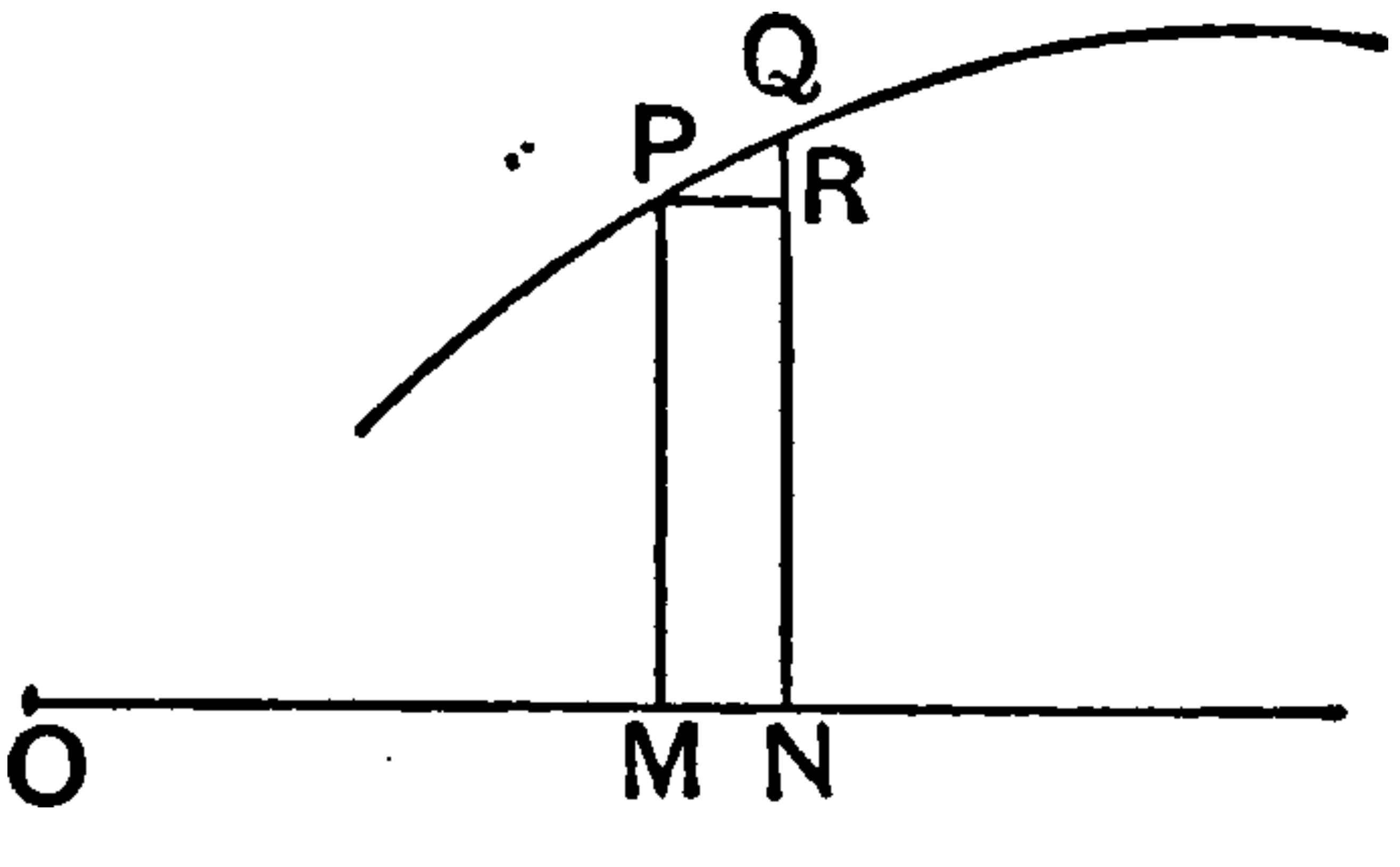
5. In the same example, show that h is a factor of $f(x+h)-f(x)$, and show to what the other factor approximates when h is very small.
Ans. $2x-3$.

6. If $\phi(x)=a^x$, prove that $\{\phi(x)\}^2=\phi(2x)$.

7. If $F(x)=\log x$, show that $F(ab)=F(a)+F(b)$.

11. The graph of a function. In any case when y is a function of x, say $y=f(x)$, the relation between x and y can be represented pictorially (as stated in Art. 6) by cutting off lengths along a straight line from a given point on it to represent the values of x, and erecting perpendiculars to represent the corresponding values of y. If the ends of these perpendiculars are joined by a curve, the curve is described as the curve $y=f(x)$ and is said to be the 'graph' of the function. The values of x cut off on the given line are called abscissae (meaning *cut-off*), the point from which they are measured is called the origin, and the perpendicular lengths are called ordinates. The line along which the values of x are measured is called the axis of x, and the line perpendicular to it drawn through the origin is called the axis of y.

12. Gradient. Differential coefficient. If $y=f(x)$, *i.e.* if y is a function of x, the ratio of the differential of y to the differential of x, or $\frac{dy}{dx}$, is called the differential coefficient of y with respect to x. If the curve $y=f(x)$ is drawn, the values of $\frac{dy}{dx}$ at the various points are the values of the tangent of the angle which the tangent to the curve makes with the axis of x, *i.e.* are the values of the 'gradient' of the curve at the various points. This is obvious from a figure, for if P, Q are two adjacent points on the curve, PQ is ultimately the tangent at P, and QR ÷ PR is the gradient of the curve. And, if $\text{OM}=x$, MN is dx; $\therefore \text{PR}=dx$, and if $\text{PM}=y$, QN − PM is dy, *i.e.* $\text{QR}=dy$; $\therefore \frac{\text{QR}}{\text{PR}}=\frac{dy}{dx}$ in the limit when both QR and PR are infinitely small.



It has been suggested, in consequence of this meaning of $\frac{dy}{dx}$ in connection with the graph of the function, that $\frac{dy}{dx}$ might always be called the gradient of y with respect to x, or, more briefly still, the x-gradient of y, instead of using the long expression, 'the differential coefficient of y with respect to x.'

13. If y is expressed as $f(x)$, its differential coefficient is often expressed as $f'(x)$, *i.e.* $\frac{dy}{dx}$ is denoted by $f'(x)$. When this notation is adopted, $f'(x)$ is often called the first derived function of $f(x)$.

It will often be convenient to draw a graph of $f'(x)$ as well as a graph of $f(x)$. The two curves have a great many interesting relations as we shall see later on. The graph of $f'(x)$ is often called the first derived curve.

Another notation sometimes used instead of $\frac{dy}{dx}$ is Dy. It is an incomplete notation, but has the advantage of brevity.

14. Summary.

If
$$y = f(x)$$
$$y + dy = f(x + dx)$$
$$\therefore dy = f(x + dx) - f(x)$$
or
$$df(x) = f(x + dx) - f(x).$$

This is the differential of y, *i.e.* its infinitely small increment when x receives an infinitely small increment dx.

The differential coefficient, or gradient, of y is the ratio of dy to dx in the limit when both are diminished to zero. It is equal to the limiting value of

$$\frac{f(x+dx) - f(x)}{dx},$$

or, as it is often put, of $\frac{f(x+h) - f(x)}{h}$,

when h (i.e. dx) $= 0$.

It is expressed in the various notations $\frac{dy}{dx}$, $f'(x)$, Dy, and sometimes even y_1 or y'. We may also write, if we please, $\frac{df(x)}{dx}$, or $Df(x)$, but these expressions are cumbrous.

The differential relation may be written $df(x) = f'(x)dx$, and this explains the reason why $f'(x)$ is called the differential coefficient, since it is the coefficient of the differential of x in the value of $df(x)$.

15. Classes of functions.

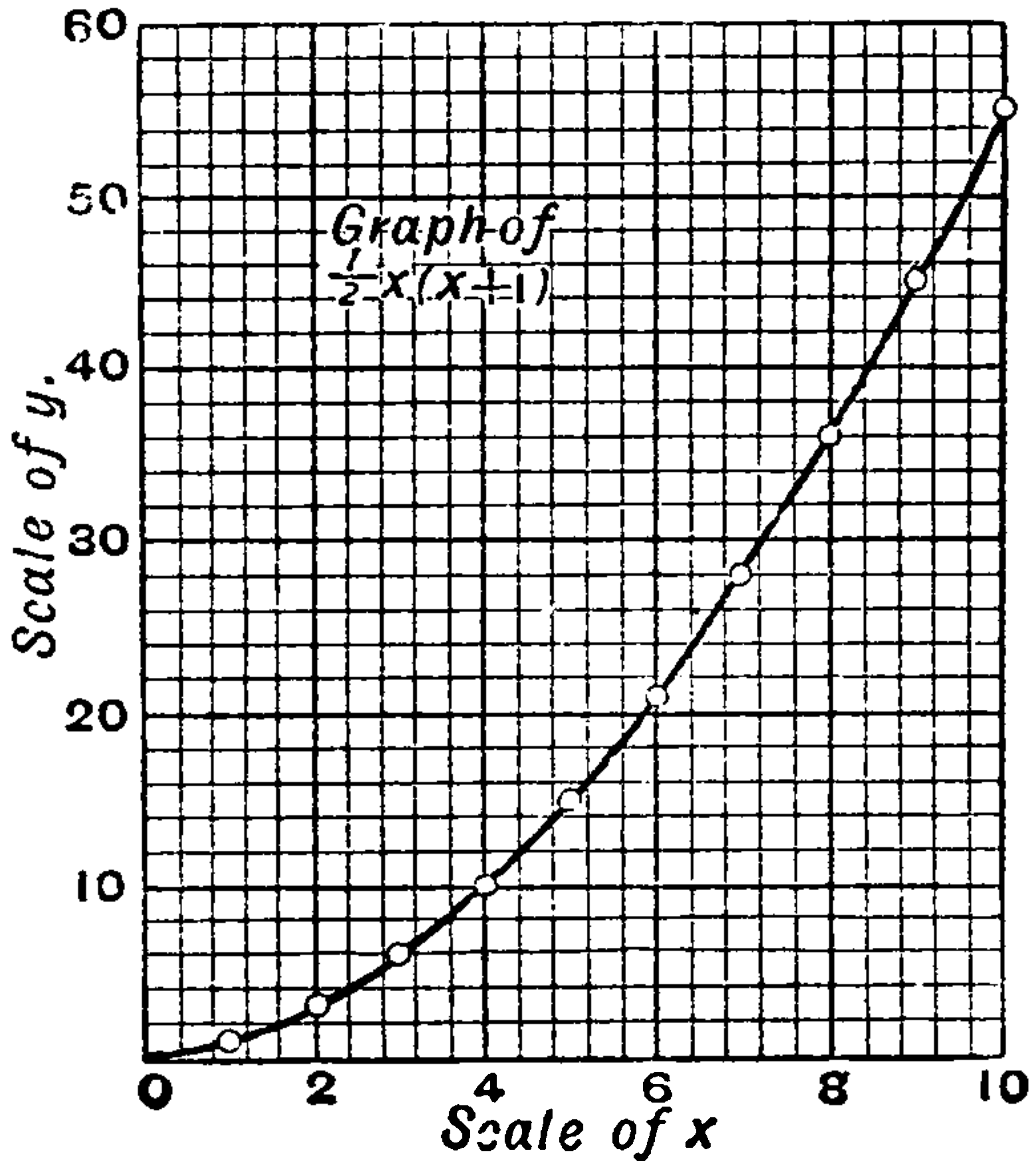
Functions of any variable x are divided into two great classes, algebraic and transcendental. Such functions as $x^2 - 3x + 2$, $\sqrt{(a^2 - x^2)}$, $\frac{3x - 7}{5x^2 - 6x + 2}$, are called algebraic, the term being used for all functions expressed in powers of x either whole or fractional, and powers and roots of such expressions. All other functions, such as $\sin x$, $\tan x$, a^x, $\log x$, $\sin^{-1}x$, are called transcendental. Many of them can be expressed in powers of x, *i.e.* algebraically, but only by means of an infinite series of such powers.

We may also divide functions into single-valued functions, and many-valued functions. For example, $x^2 - 3x + 2$ is a single-valued function of x, since there is only one value of the function for each value of x; but $\sqrt{(a^2 - x^2)}$ is a two-valued function since every number has two square roots. So also $\sin x$ is a single-valued function, but $\sin^{-1}x$ is many-valued, there being an infinite number of angles having a given sine.

There is also another important method of grouping functions, namely, into continuous and discontinuous functions. For example, the sum of the first x whole numbers is a discontinuous function, since it has no meaning when x is a fraction. Curiously enough, however, the algebraic expression which gives the sum of the first x numbers, namely $\frac{1}{2}x(x+1)$, is a continuous function of x. So that here we have a case where the graph of the algebraic expression representing the function is continuous, though only isolated points on it have reference to the actual function under consideration, viz. the sum of the first x integers. The student should draw this graph by taking different values of x, and plotting. He should make a tabular list of values, as shown, and then plot on a convenient scale.

The graph is shown in the adjoining figure.

x	$\frac{1}{2}x(x+1)$
1	1
2	3
3	6
4	10
5	15
6	21
7	28
8	36
9	45
10	55



The easiest way is to use squared paper, which can be obtained at a cheap rate from Messrs. Partridge & Cooper, Fleet Street, London, E.C., or other wholesale stationers.

Another example of a discontinuous function is the product of the first x numbers, known as *factorial* x, and denoted by $\underline{|x}$ or $x!$, whose fundamental properties are $\underline{|x} = x\underline{|x-1}$, and $\underline{|1} = 1$.

Here we can draw ordinates for each whole value of x, and we can draw a fair curve through the ends of those ordinates, but we cannot thereby interpret the meaning of the intermediate ordinates of this curve.

It is possible, however, to find an expression for factorial x which will be continuous, *i.e.* will have a value for all intermediate values of x, and which will satisfy the above fundamental laws. The curve, drawn as above, will be the graph of this continuous function, and the isolated points on it corresponding to the integer values of x will represent the product of the first x numbers. This continuous function is of great importance in some kinds of theoretical work.

Hence these two examples of apparently discontinuous functions turn out to be, in reality, examples of continuous functions, to isolated values of which we, as a rule, pay exclusive attention. There are many such functions, *e.g.* the number of spherical shot which can be piled into a pyramid with a square base is such a function.

16. In general, the functions with which we deal will be continuous functions, except for occasional isolated points of discontinuity such as when the function becomes infinite for some finite value of x, as in the case of $y = \dfrac{1}{1-x}$ of which the student might usefully draw the graph. He will find the ordinate is positive when x is less than 1, becoming infinitely great when $x = 1$; then it suddenly becomes negative, and, as x increases, y diminishes in numerical value till finally it becomes zero when x is infinitely great. There is therefore a point of discontinuity when $x = 1$, but elsewhere the curve is continuous.

But points of discontinuity in an otherwise continuous function are an entirely different thing from the discontinuity of a discontinuous function; the graph of a continuous function is a continuous curve, and its points of discontinuity are merely interruptions to the continuity of the graph, whereas the graph of a truly discontinuous function would consist of isolated points. Such would have been the graph of *factorial* x if the continuous function whose graph passes through the special points corresponding to the integer values of x had not been discovered.

EXAMPLES.

1. The sum of two numbers being 5, find the change in their product, as one of them increases from 0. Draw the graph.

2. Trace in a diagram the values of the reciprocal of a number.

3. Trace the changes in value of the sum formed by adding together a number and its reciprocal.

4. Represent in a diagram the values of 2^n for different values of n.

5. Draw the curve $y = \sin x$ from 0 to 2π. [Take horizontal distances representing the values of x, and erect ordinates proportional to the sines of these angles.]

6. Draw the curve $y = \cos x$ from 0 to 2π, and show that it is the same as the curve $y = \sin x$ from $\frac{\pi}{2}$ to $\frac{5\pi}{2}$.

7. Draw the curves $y = \tan x$, and $y = \sec x$, from 0 to 2π.

8. Draw the curve $y = \log_{10} x$.

9. Represent in a diagram the connection between the Fahrenheit and centigrade scales of temperature.

10. Represent by ordinates the values of the coefficients of the different powers of x in the expansion of $(1+x)^8$.

11. Represent in a diagram the work done by a constant force of 10 lbs. weight, as its point of application moves through varying distances in the direction of the force.

12. Represent in a diagram the force required to stretch an elastic cord whose unstretched length is 6 feet, being given that the force increases uniformly in proportion to the stretch, and that a 10 lb. weight would double the length of the cord.

13. Show that the area between the graph of the last function and the axis of x and the final ordinate represents the work done in stretching the cord. Also draw a *graph* of the work done.

14. The strength of a beam is proportional to its breadth and the square of its depth. Show the changes in the strength of a rectangular beam of given area of cross-section, as its breadth changes.

15. Trace the changes in strength of the beam with change of breadth, if the perimeter of the cross-section is constant, and equal to 20 inches.

16. A rectangular beam is cut from a cylindrical log of wood 1 foot in diameter. Trace the change in its strength according to its width. [Taking the breadth (b) and the depth (d) in inches, we have $b^2+d^2=144$, and the strength is proportional to bd^2; $\therefore$ the strength is proportional to $b(144-b^2)$. We have therefore to draw the curve $y=x(144-x^2)$ from $x=0$ to $x=12$.]

17. Find the dimensions of the above beam when the strength is a maximum. [We have to find for what value of x the ordinate of the above curve is the greatest. This will be the required breadth; and the depth can be obtained from the equation $b^2+d^2=144$.]

18. Find the dimensions of the beam of Example 15 when the strength is a maximum.

17. The last two problems are of a kind whose solution is most readily effected by the methods of the differential calculus, for evidently the maximum values required will be the ordinates of the respective graphs at points where the gradient of the graph is zero. The points can be approximately discovered by careful drawing of the graph, but they could be more rapidly found if we knew how to find an expression for the gradient at each point, and then solved the equation obtained by putting the gradient $=0$. There are many problems requiring a knowledge of the gradient of a function, or, as it is generally called, the differential coefficient of the function. The first thing, therefore, to do is to establish methods by which we can with the minimum of trouble find the differential coefficient of any function.

To do this we shall first find the differential coefficients or gradients of various simple functions, and then show how to find the gradients of more complicated expressions, which will always be found to be mere combinations of the simpler forms. The process of finding the gradient is called differentiating the function, a loose expression which is only justified by its comparative brevity. We have already discovered that, if $y=x^2$, $\frac{dy}{dx}=2x$, and have seen that we can write the result in the form $dy=2xdx$, or $d(x^2)=2xdx$; *i.e.* the *differential* of x^2 is $2xdx$, and its *differential coefficient*, or gradient, is $2x$.

Our next work will be to find the gradient in the case of any power of x, and after that, to establish general formulae by which the differentiation of more complicated functions of x is very much simplified.

18. Differentiation from first principles. We may, however, find the gradient at any point of either of the above curves without making use of any of the formulae which will be established in the next chapter, and so, by finding when the gradient is zero, solve the problems in the above examples 17 and 18. Indeed, we may in this way find independently the differential coefficient of any function we please. The method is instructive, though much more cumbrous than the systematic methods of the calculus, to be subsequently explained, by which the differential coefficients of more or less complicated functions are deduced from those of a few simple functions, independently obtained once for all.

Let (x, y) be the coordinates of any point P on the graph of question No. 16, so that $y = 144x - x^3$. Also let (x', y') be the coordinates of an adjacent point Q on the curve, so that $y' = 144x' - x'^3$. Now evidently the gradient of PQ is $\frac{y'-y}{x'-x}$, and the limiting value of this gradient when Q is brought close up to P will be the gradient of the curve at P. Now, by subtraction,

$$y' - y = 144(x' - x) - (x'^3 - x^3)$$

$$\therefore \frac{y'-y}{x'-x} = 144 - (x'^2 + x'x + x^2),$$

which in the limit (when $x' = x$) becomes equal to $144 - 3x^2$, which is therefore the value of the gradient at the point P, whose abscissa is x.

If, now, we find for what value of x this gradient is zero we shall have solved the problem in Question No. 17. The value is $x = \sqrt{48}$ or nearly 7. Hence the strongest rectangular beam which can be cut from the given log is one whose breadth is $\sqrt{48}$ inches. The corresponding depth is $\sqrt{96}$ inches, so that it is approximately 7 inches broad by 10 inches deep.

Similarly, in the case of the curve $y=x(10-x)^2$, *i.e.*

$$y=100x-20x^2+x^3$$

which is the curve of Question No. 15, we find the gradient $=100-40x+3x^2=(10-x)(10-3x)$, so that the gradient is zero when x equals either 10 or $3\frac{1}{3}$. The value of the function may be either a maximum or a minimum when its gradient is zero: on consulting the diagram, it becomes evident that the maximum strength is obtained when the breadth is $3\frac{1}{3}$ inches. The depth is then $6\frac{2}{3}$ inches, *i.e.* is twice the breadth. This is the answer to Question No. 18.

EXAMPLES.

Find, from first principles, the gradient at any point of each of the following curves:

1. $x^2+y^2=a^2$.
2. $x^2+y^2+Ax+By+C=0$.
3. $y^2=4ax$.
4. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.
5. $y=\frac{a^2}{x}$.
6. $y=x^3$, drawing the curve between $x=\pm 2$.
7. $y^2=x^3$, drawing the curve between $x=0$ and $x=2$.

CHAPTER II.

DIFFERENTIATION OF ALGEBRAIC FUNCTIONS, AND GENERAL METHODS OF DIFFERENTIATING.

19. *To find the differential of x^n.*

Let $$y = x^n,$$
then $$y + dy = (x + dx)^n$$
$$\therefore\ dy = (x + dx)^n - x^n.$$

Expanding $(x + dx)^n$ by the binomial theorem,* the first term will be x^n, which will cancel the term $-x^n$, and the second term will be $n.x^{n-1}.dx$ which is an infinitesimal of the first order, while the other terms will contain higher powers of dx, and will therefore be infinitesimals of higher orders than the first, and therefore negligible. Hence, keeping only the infinitesimal of the first order,

$$dy = nx^{n-1}dx,$$
$$\textit{i.e.}\ \mathbf{d(x^n) = nx^{n-1}dx},$$

and the gradient, or differential coefficient, is nx^{n-1}.

This is true, whatever is the value of n, so long as it is a constant.

Thus $$d(x^3) = 3x^2dx,\quad d(x^{10}) = 10x^9dx,$$
$$d(\sqrt{x}) = d(x^{\frac{1}{2}}) = \tfrac{1}{2}x^{-\frac{1}{2}}dx = \frac{dx}{2\sqrt{x}},$$
$$d(x\sqrt{x}) = d(x^{\frac{3}{2}}) = \tfrac{3}{2}x^{\frac{1}{2}}dx = \tfrac{3}{2}\sqrt{x}\,.\,dx,$$

---

* The Expansion is $(x + dx)^n = x^n + nx^{n-1}dx + \frac{n(n-1)}{1.2}x^{n-2}(dx)^2 + \dots$.

$$d\left(\frac{1}{x}\right)=d(x^{-1})=-1\,.\,x^{-2}dx=-\frac{dx}{x^2},$$

$$d\left(\frac{1}{x^3}\right)=d(x^{-3})=-3\,.\,x^{-4}dx=-\frac{3dx}{x^4},$$

and so on.

EXAMPLES.

1. Find the differential coefficients of the above functions from first principles (see Art. 18). In the case of $\sqrt{x}$, having obtained

$$\frac{y'-y}{x'-x}=\frac{\sqrt{x'}-\sqrt{x}}{x'-x}$$

we have to divide numerator and denominator by $\sqrt{x'}-\sqrt{x}$, and then simplify by putting $x'=x$ in the result. Similarly in the case of $x\sqrt{x}$.

2. Find the differential coefficient of $x^{\frac{3}{5}}$ from first principles.

[NOTE. The fraction to be simplified may be written in the form $\frac{z'^3-z^3}{z'^5-z^5}$, where $z=x^{\frac{1}{5}}$. Then it is obvious that $z'-z$ is the factor to be cancelled.]

3. Find the differential coefficient of $x^{\frac{p}{q}}$ from first principles, where p and q are positive integers; and deduce the differential coefficient of x^{-n}, where n is any positive rational fraction.

20. Before going on to other functions it will be useful to see how far-reaching the use of this formula for the differential of x^n is, and how many difficult expressions can be differentiated by means of it, with the help of the following general theorems which are applicable to all functions.

(i) If u is any function of x, and a is any constant, the differential of $au=a.du$, *i.e.* $d(au)=a du$.

For, if $$y=au,$$

$$y+dy=a(u+du);$$

$\therefore$ by subtraction, $$dy=a.du,$$

i.e. $$\mathbf{d(au)=a\,du.}$$

For example, $d(\pi r^2)=\pi d(r^2)$, as we have already seen.

(ii) If u, v, w are any functions of x, the differential of their sum is equal to the sum of their separate differentials,

i.e. $$\mathbf{d(u+v+w)=du+dv+dw.}$$

This theorem is self-evident, and both it and the preceding theorem would be equally true if the increments of the functions were not infinitely small.

(iii) The differential of a constant is $=0$. This also is obvious, as it merely expresses in the language of the calculus that the quantity *is* constant.

Thus, if a is a constant,

$$\mathbf{d(a)=0.}$$

NOTE.—Generally, if we use letters to denote constants, and also letters to denote variables, we shall use the early letters of the alphabet, such as a, b, c, or l, m, n to denote constants, and the later letters such as x, y, z or u, v, w to denote variables. This is merely for convenience, in order not to have to state always which quantities are variable, and which are constant.

Example. If $y=2x^3-3x^2+5x-7$

$$dy=2d(x^3)-3d(x^2)+5dx-0$$

by the theorems just established.

Hence, since $dx^n=nx^{n-1}dx$, it follows that

$$dy=6x^2dx-6x\,dx+5dx$$
$$=(6x^2-6x+5)dx\,;$$
$$\therefore\ \frac{dy}{dx}=6x^2-6x+5.$$

We will now take a harder example, first premising that the use of x as the principal variable is merely a matter of convenience in establishing our theorems. Any other variable would do equally well, *e.g.* $d(u^n)=nu^{n-1}du$ where u may be any variable, and may itself be a function of some principal variable x.

Example. Find the x-gradient of $(3x^2-2x+5)^3$.

Let $$y=(3x^2-2x+5)^3,$$

then, treating the expression in brackets as a single quantity, we have

$$dy=3(3x^2-2x+5)^2\,.\,d(3x^2-2x+5),$$

on the principle that $d(z^3)=3z^2\,dz$, where $z=3x^2-2x+5$;

and now, simplifying the last factor, we have finally

$$dy=3(3x^2-2x+5)^2(6x\,dx-2dx)$$

and $$\frac{dy}{dx}=3(3x^2-2x+5)^2(6x-2)$$
$$=6(3x-1)(3x^2-2x+5)^2.$$

We could have differentiated this expression by first cubing out, and then differentiating each term separately, but the process would have been intolerably long, and the form of the result would have been less useful than the factorized form obtained by the simple artifice of initially treating $3x^2-2x+5$ as a single quantity.

Again. Find the differential coefficient of $\sqrt{(3x^2-2x+5)}$.

Let $$y=\sqrt{(3x^2-2x+5)}.$$

Now $$d(\sqrt{z})=d(z^{\frac{1}{2}})=\frac{dz}{2\sqrt{z}};$$

$$\therefore\ dy=\frac{d(3x^2-2x+5)}{2\sqrt{(3x^2-2x+5)}}$$
$$=\frac{(6x-2)\,dx}{2\sqrt{(3x^2-2x+5)}},$$

in which 2 is a factor which can be cancelled; and, finally,

$$\frac{dy}{dx}=\frac{3x-1}{\sqrt{(3x^2-2x+5)}}.$$

21. In the following examples we shall sometimes denote a function by y, and sometimes by $f(x)$. The corresponding notations for the gradient of the function are, as before stated, $\frac{dy}{dx}$ or Dy, and $f'(x)$, respectively. The student may in every case use which notation he prefers, but he will probably find it best to *work* with the single letter y, and then to please himself as to the form in which he states the final result.

EXAMPLES.

1. If $y=x(10-x)^2$, find Dy, and find for what points the gradient is zero. [See Examples 15, 18, p. 16.]

2. If $f(x)=x(144-x^2)$, find $f'(x)$, and find the values of x for which $f'(x)=0$. [See Examples 16, 17, p. 16.]

3. If $f(x)=3\sqrt{x}-5x\sqrt{x}+\frac{2}{\sqrt{x}}$, find $f'(x)$.

4. If $y=\frac{x^n}{\lfloor n}$, prove that $\frac{dy}{dx}=\frac{x^{n-1}}{\lfloor n-1}$.

5. If $y=a_0+a_1x+a_2\frac{x^2}{\lfloor 2}+a_3\frac{x^3}{\lfloor 3}+\dots$, prove that

$$Dy=a_1+a_2x+a_3\frac{x^2}{\lfloor 2}+\dots,$$

6. Draw the curve $y=x^3$, and also its first derived curve, between the limits $x=\pm 2$.

7. If $y^2=x^3$, find Dy in terms of x.

Draw the curve between $x=0$ and $x=4$, showing carefully the slope of the curve at the points corresponding to these extreme values of x.

8. Draw the curve $y=2x^3-3x^2+7x-2$ between the limits $x=-2$ and $x=+3$, taking the vertical scale $=\frac{1}{10}$ of the horizontal scale.

9. Draw the curve $y=6x^2-6x+7$, using the same vertical and horizontal scales as in No. 8.

10. If $f(x)=(x^2-2x)^2$, find $f'(x)$.

Draw the graph between $x=-1$ and $x=3$, finding its gradient at the points where $x=-1, 0, 1, 2, 3$ respectively, *i.e.*, find $f'(-1)$, $f'(0)$, $f'(1)$, $f'(2)$, and $f'(3)$. Draw also the graph of $f'(x)$.

11. Find Dy, when $y=\frac{1}{x}-\frac{1}{2x^2}+\frac{1}{3x^3}$.

12. If $4x^2+9y^2=36$, find $\frac{dy}{dx}$.

13. A ball is thrown up with a velocity of 50 feet per second from the top of a tower 100 feet high. Its height above the ground in t seconds is $y=100+50t-16t^2$. Plot the curve of displacement from the start to when the ground is reached.

14. In the equation of last question find $\frac{dy}{dt}$, which will be the velocity at each instant, and find at what time the velocity is zero (when the ball will be at its highest point). Plot the diagram of velocity, viz., $y_1=\frac{dy}{dt}$, taking values of t as abscissae and values of y_1 as ordinates.

15. Find $\frac{dy}{dx}$ in the following cases:—

(1) $y=(3x^2-5x+6)^{-2}$.

(2) $y=\frac{2}{x^2-6}-\frac{2}{x^2+6}$.

(3) $y=\sqrt{(x^2-3x+2)}$.

(4) $y=\frac{1}{x+\sqrt{(x^2-1)}}$.

(5) $3y^{\frac{2}{3}}+5x^{\frac{2}{3}}=15$.

16. Show that the graph of an equation of the first degree in x and y is a straight line.

[Take the equation $Ax+By+C=0$, and show that $\frac{dy}{dx}$ is the same for every point.]

Draw the line $2x-3y+6=0$, and find what its gradient is.

17. Find the gradient of the parabola $y=Ax^2+Bx+C$ at the point where $x=p$, and show that it is equal to the gradient of the chord joining the points on the curve where $x=p\pm q$.

22. The only algebraic expressions which at present we have not learned how to differentiate systematically are *products*, such as $(2x^2-3)(5x-7)^3$, and quotients, such as $\frac{2x^2-3}{5x-7}$. We could find the differential of the former by multiplying out and then differentiating each term separately, but it would be a laborious task, and this would moreover be rendered more difficult if any of the factors were raised to fractional powers. A quotient could not without very heavy work be differentiated except by one or other of the methods now to be explained, which are therefore of great importance.

23. Product. If u, v are two functions of a single variable x, find the differential of uv.

Let $$y=uv.$$

Let x receive an infinitesimal increment dx, causing u and v to receive the respective increments du and dv. Then the increased value of the product is given by the equation

$$\begin{aligned} y+dy &= (u+du)(v+dv)\,; \\ \therefore\quad dy &= (u+du)(v+dv)-uv \\ &= vdu+udv+du\,.\,dv \\ &= vdu+udv \end{aligned}$$

in the limit, since $du\,.\,dv$ is an infinitesimal of the second order being the product of two infinitely small quantities. Hence

$$\mathbf{d(uv)=v\,du+u\,dv.}$$

24. This important rule is the rule for finding the differential of a product. The student should observe that if v is a constant, $d(uv)=vdu$, which is in accordance with Rule (i) Art. 20, and similarly, if u is a constant, $d(uv)=udv$. Hence the complete differential of the product when u and v both vary is the sum of the partial differentials obtained on the supposition that first one and then the other of the two factors is constant.

If we divide by dx, which we have supposed to cause the increments of the functions, we have the equation between the differential coefficients, viz.:

$$\frac{d(uv)}{dx}=v\frac{du}{dx}+u\frac{dv}{dx}.$$

which may be written

$$\mathbf{D(uv)=v\,Du+u\,Dv.}$$

For example, let $y=(2x^2-3)(5x-7)^3.$

Here $du=4x\,dx$;

$\therefore\ Du=4x,$

and $dv=3(5x-7)^2d(5x-7)$

$=15(5x-7)^2dx$

$\therefore\ Dv=15(5x-7)^2$;

$\therefore\ Dy=(5x-7)^3.4x+(2x^2-3).15(5x-7)^2$

$=(5x-7)^2\{20x^2-28x+30x^2-45\}$

$=(5x-7)^2(50x^2-28x-45).$

25. The student should carefully go through this work, and compare it with the result of multiplying out the factors of y and *then* differentiating. Not only would this latter method be much longer, but it would lose the great advantage of showing that $(5x-7)^2$ is a factor of the result, which is an important fact worthy of attention. Indeed, always, if any factor in the expression for y occurs raised to the n^{th} power, the same factor will occur also in $\frac{dy}{dx}$, raised to the $n-1^{\text{th}}$ power. (See Ex. 4, below.)

EXAMPLES.

Find $\frac{dy}{dx}$ in the following cases:

1. $y=(x-3)^4(2x-7)^5.$

Ans. $(x-3)^3(2x-7)^4(18x-58).$

2. $y=(3x^2-7x+2)^{\frac{3}{2}}(x^2-7)^{\frac{5}{2}}.$

Ans. $\frac{1}{2}(3x^2-7x+2)^{\frac{1}{2}}(x^2-7)^{\frac{3}{2}}(48x^3-91x^2-106x+147).$

3. $y=(x^2-2)(7x-3)^{-1}$.

Ans. $(7x^2-6x+14)(7x-3)^{-2}$.

4. $y=u^m v^n$.

Ans. $u^{m-1}v^{n-1}(mv\,Du+nu\,Dv)$.

5. If a volume v of a gas contained in a vessel under pressure p is compressed or expanded without loss of heat, the law connecting v and p is that $p.v^k$ is constant. Prove that for a slight compression or expansion the connection between the changes of volume and pressure is given by the equation

$$kpdv+vdp=0.$$

26. If y consists of more than two factors, the preceding method can be used by grouping the factors into two groups: thus

Let $y=uvw=u\,.\,vw$

then
$$\begin{aligned} dy &= vw\,.\,du+u\,.\,dvw \\ &= vw\,.\,du+u(wdv+vdw) \\ &= vwdu+uwdv+uvdw, \end{aligned}$$

and $Dy=vwDu+uwDv+uvDw$;

and similarly in the case of more than three factors.

Of course the labour of differentiating in such a case is necessarily heavy, however systematically it may be done. Indeed in the case of simple factors, not repeated, it is probably better to multiply out first and then differentiate.

The student might test this in the cases of $(x-1)(x-2)(x-3)$ and $(x-1)(x-2)(x-3)(x-4)$.

27. Quotient. If $y=\dfrac{u}{v}$

$$y+dy=\frac{u+du}{v+dv}$$

$$\begin{aligned} \therefore\ dy &= \frac{u+du}{v+dv}-\frac{u}{v} \\ &= \frac{vdu-udv}{v(v+dv)} \\ &= \frac{vdu-udv}{v^2}, \text{ in the limit.} \end{aligned}$$

i.e.
$$\mathbf{d}\left(\frac{\mathbf{u}}{\mathbf{v}}\right)=\frac{\mathbf{v\,du-u\,dv}}{\mathbf{v^2}}$$

and
$$\frac{d\left(\frac{u}{v}\right)}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2},$$

which may be written
$$\mathbf{D}\left(\frac{\mathbf{u}}{\mathbf{v}}\right)=\frac{\mathbf{v\,Du-u\,Dv}}{\mathbf{v^2}}.$$

The student should also obtain this result by writing the fraction in the form uv^{-1} and differentiating by the rule for differentiating a product.

Example. Let
$$y=\frac{2x^2-3}{5x-7};$$
$$\therefore \frac{dy}{dx}=\frac{(5x-7).4x-(2x^2-3).5}{(5x-7)^2}$$
$$=\frac{10x^2-28x+15}{(5x-7)^2}.$$

Again, let
$$y=\frac{2x^2-3}{(5x-7)^3}.$$

This is most conveniently differentiated as a product, writing it in the form
$$y=(2x^2-3)(5x-7)^{-3};$$
$$\therefore \frac{dy}{dx}=4x(5x-7)^{-3}-15(2x^2-3)(5x-7)^{-4}$$
$$=\frac{4x(5x-7)-15(2x^2-3)}{(5x-7)^4}$$
$$=\frac{45-28x-10x^2}{(5x-7)^4}.$$

The student should work the same example by the Quotient formula. He will find that a common factor, $(5x-7)^2$, must be cancelled from numerator and denominator before the result is expressed in its simplest form.

28. Frequently the student might find it advantageous, if a factor is repeated, in the case of *either* a product *or* a quotient, to use the easily proved formula
$$D(u^m v^n)=u^{m-1}v^{n-1}(mv\,Du+nu\,Dv).$$

This was given as an example to be proved in No. 4 of the last set of examples.

29. The student ought now to be able to differentiate any algebraic expression whatever. He cannot yet differentiate transcendental expressions, though in many cases he can do something towards it.

E.g. we know that $(\sin x)^2+(\cos x)^2=1$

$$\therefore\ 2\sin x.d(\sin x)+2\cos x.d(\cos x)=0.$$

This step is obtained by treating $\sin x$ and $\cos x$ as single quantities, like $u^2+v^2=1$.

We do not yet know how to express $d(\sin x)$ and $d(\cos x)$ in terms of dx. This and other formulae will be the subject of next chapter. Meanwhile we may show one class of problems which may be solved by the help of differentials.

30. Approximations by help of differentials.

When a variable quantity x receives an actual increment, the approximate change in any given function of x is given by the value of its differential, provided the change in x is small.

Thus, to find $(40{\cdot}1)^2$ we can use the formula $(x+dx)^2=x^2+2x\,dx$ or $(x+h)^2=x^2+2xh$, which only differs from the true value by h^2.

Hence $(40{\cdot}1)^2=1600+8=1608$, approximately. The true value is $1608{\cdot}01$.

Again, the area of a triangle having two sides a, b, and included angle C, is $\frac{1}{2}ab\sin \mathrm{C}$.

If each side is increased by a short length h, the angle being unchanged, the increase in area is approximately $\frac{1}{2}(adb+bda)\sin \mathrm{C}$, where da and db are each equal to h, *i.e.* the increment of area $=\frac{1}{2}h(a+b)\sin \mathrm{C}$, approximately.

Again, evaluate

$$\frac{403}{1728}-\frac{400}{1725}.$$

From the formula $$d\left(\frac{u}{v}\right)=\frac{v\,du-u\,dv}{v^2}$$

we have, approximately, the required difference

$$=\frac{3(1725-400)}{(1725)^2}=\frac{3975}{(1725)^2}.$$

The true difference is $$\frac{3975}{1725\times 1728}.$$

Again, evaluate $3x^2 - 7x + 5$ when $x = 3{\cdot}02$.

We have, approximately, $f(x+dx) = f(x) + f'(x)dx$

$$= 3x^2 - 7x + 5 + (6x - 7)dx.$$

$\therefore$ if we take $x=3$, and $dx = {\cdot}02$, we find $f(3{\cdot}02) = 11 + 11 \times {\cdot}02 = 11{\cdot}22$ approximately. The true value is $11{\cdot}2212$.

Such examples may be multiplied indefinitely.

We shall use this method in Chapter VI. to approximate to the roots of equations.

NOTE.—If the student prefers to restrict the use of dx to the case when the increment of x is infinitely small, he can use a single letter such as h, above, for a finite increment, or he may use δx for the finite increment.

Similarly k, or δy, may be used for a finite increment of a variable y, the term dy being kept for the infinitely small or vanishing increment. But in practice the distinction is unnecessary so long as the increments δx and δy are small enough for their squares and higher powers to be negligible.

31. Recapitulation.

$$d(\text{constant}) = 0,$$

$$d(au) = a\,du,$$

$$d(u + v + w + \,.\,.\,) = du + dv + dw + \ldots,$$

$$d(uv) = v\,du + u\,dv,$$

$$d\frac{u}{v} = \frac{v\,du - u\,dv}{v^2},$$

$$d(u^n) = nu^{n-1}du.$$

Additional formula, sometimes useful,

$$\frac{d}{dx}(u^m v^n) = u^{m-1}v^{n-1}\left(mv\frac{du}{dx} + nu\frac{dv}{dx}\right),$$

when m and n are positive or negative constants.

NOTE.—In any of the above, if we wish to use differentials, we employ the letter d; if we wish to express differential coefficients, we write either D, or d divided by dx, as shown in the last formula above.

EXAMPLES.

1. Find $\frac{dy}{dx}$ in the following cases:

(1) $y=\frac{x+a}{x-a}$.

(2) $y=(x+a)\sqrt{(x-a)}$.

(3) $y=\sqrt{\frac{x+a}{x-a}}$.

(4) $y=(1+x^2)(1+4x^2+3x^4)$.

(5) $y=\frac{(1+x^2)^3}{(1+x^3)^2}$.

(6) $y=\frac{x}{\sqrt{(a^2-x^2)}}$.

2. If two functions of x are identical, their differential coefficients are also identical. The student may illustrate this by the following two examples:

(1) Differentiate both sides of the identity

$$(a+x)^n=a^n+na^{n-1}x+\frac{n(n-1)}{1.2}a^{n-2}x^2+\dots+x^n.$$

Do the same also with a special value of n, say $n=4$.

(2) Prove the rule for differentiating a product by making use of the important identity

$$uv=\tfrac{1}{4}(u+v)^2-\tfrac{1}{4}(u-v)^2,$$

and assuming the rule for differentiating a square.

3. Differentiate $\frac{1}{a+x}$.

4. Differentiate the result of (3), and then again the new result, and so on four times.

5. Evaluate (approximately) $x^3-4x^2-11x+32$, when $x=10{\cdot}32$.

6. Evaluate the same function, when $x=2\tfrac{2}{15}$.

7. Differentiate $y=x\sqrt{\frac{x^2+a^2}{x^2-a^2}}$.

8. Differentiate $(x-3)^2(x+1)^3(2x-7)$.

9. Find $\frac{dy}{dx}$ when $x^2y^2=a^2(x^2-y^2)$.

10. Find $\frac{dy}{dx}$ when $y=f(z)$ and $z=\phi(x)$.

11. Find $\frac{dy}{dx}$ when $y=(3z^2-2z+5)^{\frac{3}{2}}$ and $z=4x^2-8x+7$.

12. Find $\frac{dy}{dx}$ when $y=\frac{3z}{z^2-5}$, and $z=\frac{3x}{x^2-5}$.

CHAPTER III.

DIFFERENTIATION OF THE CIRCULAR FUNCTIONS AND THE INVERSE CIRCULAR FUNCTIONS.

32. Before finding the differentials and differential coefficients of these functions, it is necessary first to establish the important, though simple, theorem that

When an angle is very small, its sine and tangent are practically equal to each other and to the circular measure of the angle. By being equal, is meant that their ratio is ultimately one of equality. (See Art. 2.)

It is easy to prove that $\frac{\sin\theta}{\tan\theta}=1$ when θ is ultimately made infinitely small.

For, always, $\frac{\sin\theta}{\tan\theta}=\cos\theta$, and, when $\theta=0$, $\cos\theta=1$, which proves the proposition.

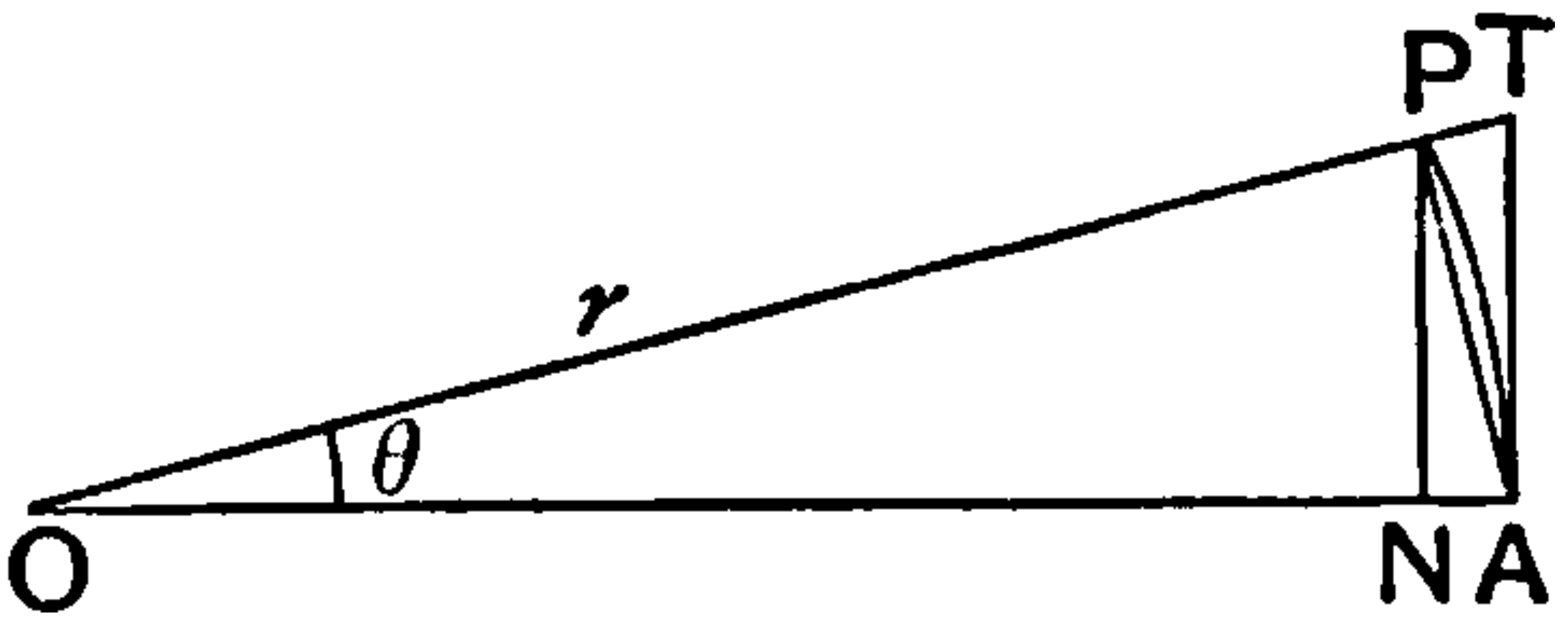
With regard to the ultimate equality of θ with both $\sin\theta$ and $\tan\theta$, a little consideration will convince the student that the circular measure of a small angle θ is intermediate in value between $\sin\theta$ and $\tan\theta$, and therefore when $\sin\theta=\tan\theta$, *i.e.*, when $\sin\theta:\tan\theta=1$ by reason of the smallness of θ, it follows that θ is equal to both of them. If a student turns to a table of values, he will see that these three quantities are very nearly equal when the angle is as much as 1° or even more.

The theorem just proved may be rendered clearer by a figure.

Let POA be a small angle whose circular measure $=\theta$. Then if with O as centre and radius r an arc PA is drawn, cutting the arms of the angle in P and A respectively, $\theta=\frac{\text{arc PA}}{r}$; and if PN is drawn perpendicular to OA to meet it in N, and AT is drawn perpendicular to OA to meet OP produced in T,

$$\sin\theta = \frac{\text{PN}}{r}, \text{ and } \tan\theta = \frac{\text{TA}}{r};$$

$$\therefore \sin\theta : \theta : \tan\theta = \text{PN} : \text{arc PA} : \text{TA}.$$



Now make θ infinitely small, which, for our purpose, is best done by making r infinitely great, while T remains fixed. We see that PN, the arc PA, and TA ultimately coincide, so that they are ultimately equal. Therefore $\sin\theta$, θ, and $\tan\theta$, which are proportional to them, also become ultimately equal, which again establishes the required proposition.

Cor. This shows that, to the first order of small quantities, the lines PN, TA, and the chord PA, and the arc PA may all be treated as equal to each other, when the angle is small. This will be a great help in the geometrical treatment of problems dealing with small angles. We also see that ultimately the chord PA is perpendicular to both the radii OP and OA, being indistinguishable from the arc PA.

33. *To find the differential of* $\sin x$; i.e., *if the angle* x *receives a small increment,* dx, *find what is the corresponding increment in the sine of the angle.*

Let $y = \sin x$,

then $y + dy = \sin(x + dx)$,

$$\therefore dy = \sin(x+dx) - \sin x,$$
$$= 2\cos(x + \tfrac{1}{2}dx)\sin(\tfrac{1}{2}dx),$$

from the formula

$$\sin \text{A} - \sin \text{B} = 2\cos\frac{\text{A}+\text{B}}{2}\sin\frac{\text{A}-\text{B}}{2}.$$

But $\sin\frac{1}{2}dx$ is ultimately equal to $\frac{1}{2}dx$, by the above theorem.

if the angle is in circular measure. Also $\cos(x+\frac{1}{2}dx)$ is ultimately equal to $\cos x$;

$\therefore$ finally $$dy = 2\cos x.\tfrac{1}{2}dx = \cos x.dx,$$

$$i.e.\ \mathbf{d\sin x = \cos x.dx}$$

and $$\mathbf{\frac{d\sin x}{dx} = \cos x.}$$

34. *To find the differential of* $\cos x$.

We will establish this in several ways, each of which is instructive.

(1) Let $$y = \cos x,$$

then $$\begin{aligned} dy &= \cos(x+dx) - \cos x, \\ &= -2\sin(x+\tfrac{1}{2}dx).\sin\tfrac{1}{2}dx, \\ &= -\sin x\, dx, \end{aligned}$$

$$i.e.\ \mathbf{d\cos x = -\sin x\, dx.}$$

This first method is differentiation from first principles. The other two methods are based on previous theorems.

(2) Let $$y = \cos x,$$

$$\therefore\ y = \sin\left(\frac{\pi}{2} - x\right),$$

$$\therefore\ dy = \cos\left(\frac{\pi}{2} - x\right)\ d\left(\frac{\pi}{2} - x\right),$$ by the formula of last article,

$$\begin{aligned} &= \sin x.d\left(\frac{\pi}{2} - x\right) \\ &= -\sin x.dx. \end{aligned}$$

(3) $$\cos^2 x = 1 - \text{siu}^2 x,$$

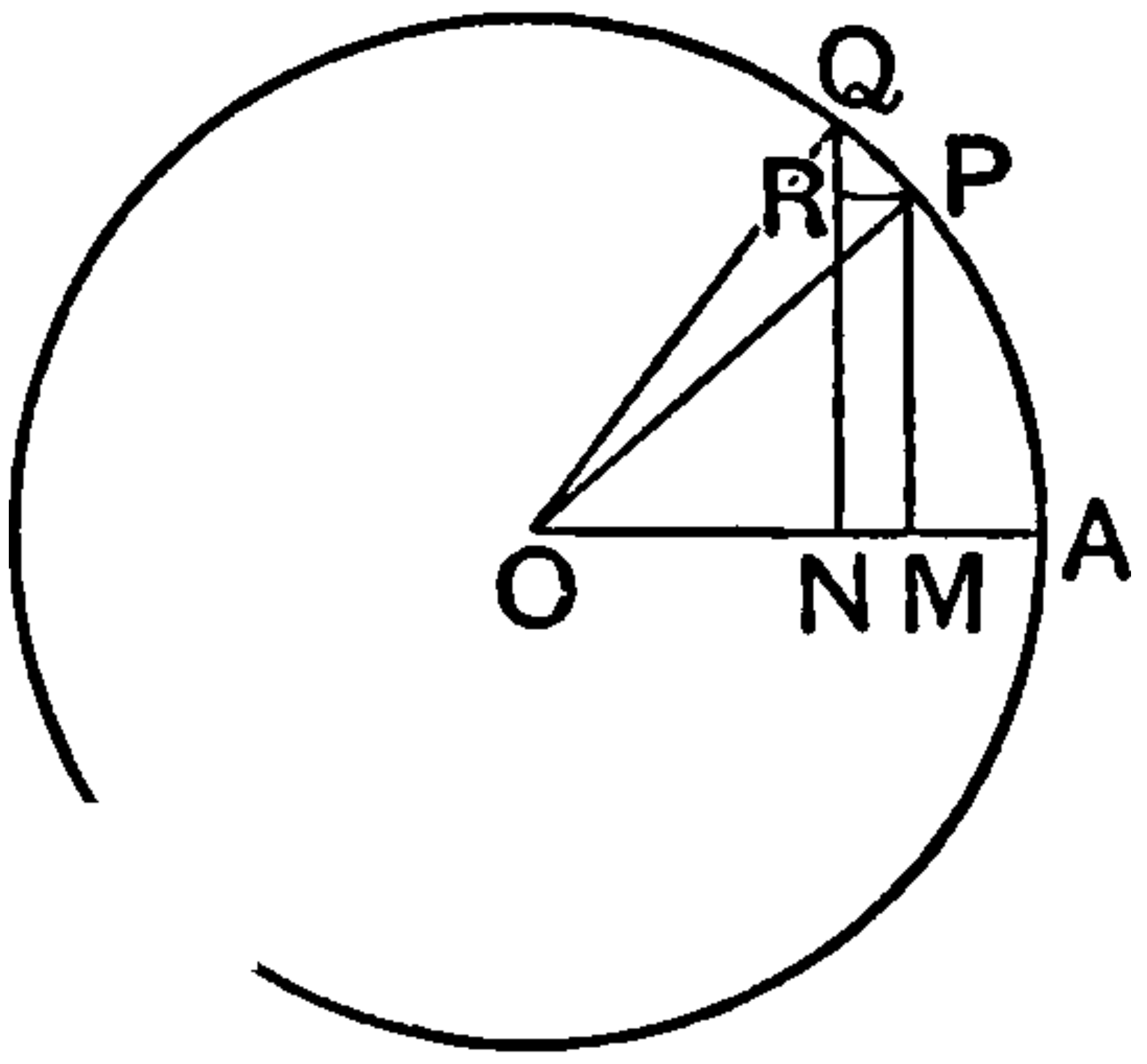
$$\begin{aligned} \therefore\ 2\cos x\, d\cos x &= -2\sin x.d\sin x, \\ &= -2\sin x.\cos x.dx, \text{ by last article,} \end{aligned}$$

$$\therefore\ d\cos x = -\sin x\, dx, \text{ as before.}$$

Expressing this in the form of a differential coefficient, we have

$$\mathbf{\frac{d\cos x}{dx} = -\sin x.}$$

35. The formulae for the differentials of the sine and cosine are so important that we will also prove them geometrically. The student should draw his own figure and carefully go through every part of the work, as it is important that he should be able to deal geometrically with infinitely small quantities. In the figure the increments have to be made of a finite magnitude, but they must be imagined to be infinitely small.



Let $\text{POA} = x,$

and $\text{QOA} = x + dx,$

so that $\text{QOP} = dx.$

If the angles are in circular measure, $dx = \dfrac{\text{QP}}{r}.$

Now $$\sin x = \frac{\text{PM}}{r}, \quad \sin(x + dx) = \frac{\text{QN}}{r};$$

$$\therefore \quad d \sin x = \frac{\text{QN} - \text{PM}}{r} = \frac{\text{QR}}{r}$$

$$= \frac{\text{QR}}{\text{QP}} \cdot \frac{\text{QP}}{r} = \frac{\text{QR}}{\text{QP}} \cdot dx.$$

Now $\dfrac{\text{QR}}{\text{QP}}$ ultimately becomes $= \cos \text{NQP}$, and this angle becomes equal to the angle at O, $= x$;

$$\therefore \quad \frac{\text{QR}}{\text{QP}} \text{ ultimately equals } \cos x,$$

and, therefore, $d \sin x = \cos x \,.\, dx.$

Similarly, $d \cos x = \dfrac{\text{ON} - \text{OM}}{r} = -\dfrac{\text{MN}}{r} = -\dfrac{\text{RP}}{r}$

$$= -\frac{\text{RP}}{\text{QP}} \cdot \frac{\text{QP}}{r} = -\,\text{siu}\, x \,.\, dx.$$

Interesting examples illustrating the use of the differential coefficients of the sine and cosine of an angle will be found in the articles on uniform circular motion and simple harmonic motion at the end of this chapter. (Articles 57-60.)

36. Gradients of the curves $y=\sin x$ and $y=\cos x$. The gradient of the curve $y=\sin x$ is given by the equation $\frac{dy}{dx}=\cos x$. This means that the numerical value of the gradient $=\cos x$, *i.e.* $=\cos x$ measured in vertical units $\div$ 1 on the horizontal scale. Moreover, we only proved that $\frac{dy}{dx}=\cos x$ on the assumption that the angles were measured in circular measure, hence 1 on the horizontal scale must mean 1 radian, which in degrees is $180^\circ \div \pi$, *i.e.* $57\frac{1}{4}$ degrees, approximately. If therefore we turn to the graph of $\sin x$, we shall be able to construct the tangent at each point P by drawing PU horizontally to the right, of length representing $57\frac{1}{4}^\circ$ on the horizontal scale, and erecting a perpendicular UT of length equal to $\cos x$ on the vertical scale (drawing UT upwards if $\cos x$ is positive, downwards if $\cos x$ is negative): PT will be the tangent at P. The construction is left to the student, who should verify its truth by drawing the graph of siu x (from 0 to π will be sufficient) on a large scale, and actually constructing the tangent by the above method at several points of the graph. By this means he will get a geometrical feeling of the truth of the theorem that

$$\frac{d\sin x}{dx}=\cos x.$$

He may also obtain a numerical illustration of the truth of the theorem by taking two angles A, B nearly equal, and forming the fraction whose numerator is $\sin \mathrm{B} - \sin \mathrm{A}$, and whose denominator is the circular measure of $\mathrm{B} - \mathrm{A}$. On dividing out, so as to express the quotient as a decimal, he will find it is equal to the cosine of some angle intermediate between A and B.

Take, for example, $\mathrm{A}=37^\circ\ 20'$ and $\mathrm{B}=37^\circ\ 21'$, so that the

$$\text{fraction}=\frac{\sin 37^\circ\ 21' - \sin 37^\circ\ 20'}{\text{circular measure of } 1'}=\frac{\cdot 0002313}{\cdot 0002909}=\cdot 7950.$$

We can obtain the quotient to only four figures correctly, even if we have used a seven-figure table, since the numerator and denominator do not contain more than four significant figures, but this degree of accuracy is enough to show that the quotient lies between the cosines of A and B, which are ·7951 ... and ·7949 .... Of course, our fraction gives the gradient of the *chord* joining the two given points, and this is obviously parallel to the tangent at some point of the graph intermediate between these two points. We have therefore verified that the gradient at this intermediate point is numerically equal to the cosine of an intermediate angle.

Similarly we might verify that the gradient of the chord joining two points on the cosine graph is numerically equal to the sine of an intermediate angle, but is *negative* (if the angles are acute) because the cosine gets smaller as the angle increases, and therefore the fraction $\dfrac{\cos B - \cos A}{B - A}$ is negative.

37. If we combine the new formulae with the results of the preceding chapter, we shall be able to differentiate any function containing sines and cosines and algebraic expressions.

For example, find the differential coefficient of $\sin^2 x \cos x$.

$$\text{Let} \qquad y = \sin^2 x \cos x\,;$$

$$\begin{aligned} \therefore\ dy &= \cos x . d(\sin^2 x) + \sin^2 x . d(\cos x) \\ &= \cos x . 2 \sin x\, d(\sin x) + \sin^2 x(-\sin x\, dx) \\ &= 2 \cos^2 x \sin x\, dx - \sin^3 x\, dx\,; \end{aligned}$$

$$\therefore\ \frac{dy}{dx} = \sin x(2 \cos^2 x - \sin^2 x).$$

An interesting illustration of the formulae will be furnished by taking an identity such as $\sin 3x = 3 \sin x - 4 \sin^3 x$ and differentiating both sides. Since the two sides are *identically* equal, it follows that their differentials must be equal, and therefore also their differential coefficients must be equal, and the relation between them will form a new identity. Thus:

$$\sin 3x = 3 \sin x - 4 \sin^3 x\,;$$

$$\therefore\ \cos 3x\, d(3x) = 3 \cos x\, dx - 12 \sin^2 x \cos x\, dx\,;$$

$$\therefore\ 3\cos 3x\,dx = 3(\cos x - 4\sin^2 x\cos x)\,dx;$$

$$\therefore\ \cos 3x = \cos x - 4\sin^2 x\cos x$$

$$= 4\cos^3 x - 3\cos x.$$

Examples for practice will be found at the end of this chapter. We will now proceed to find the differentials of the other circular functions and of the inverse circular functions.

38. *To find the differential of* $\tan x$.

Let $y = \tan x;$

$$\therefore\ dy = \tan(x + dx) - \tan x$$

$$= \frac{\tan x + \tan dx}{1 - \tan x \tan dx} - \tan x$$

$$= \frac{(1 + \tan^2 x)\tan dx}{1 - \tan x \tan dx}.$$

Now, so far, we have made no use of the infinite smallness of dx. When we introduce this simplification, we may simplify the numerator by writing dx for $\tan dx$ (Art. 32), and the denominator is ultimately equal to 1. (Art. 2.)

$$\therefore\ \text{finally,}\quad dy = (1 + \tan^2 x)\,dx$$

$$= \sec^2 x\,dx;$$

i.e. $\mathbf{d\,(tan\,x) = sec^2x\,dx}$

and $\mathbf{\dfrac{d\,tan\,x}{dx} = sec^2x.}$

Modification of the above method.

We might proceed as follows:—

$$dy = \tan(x + dx) - \tan x$$

$$= \frac{\sin(x + dx)}{\cos(x + dx)} - \frac{\sin x}{\cos x}$$

$$= \frac{\sin(x + dx)\cos x - \cos(x + dx)\sin x}{\cos(x + dx)\cos x}$$

$$=\frac{\sin dx}{\cos (x+dx)\cos x}$$

$$=\frac{dx}{\cos^2 x}, \text{ in the limit,}$$

$$=\sec^2 x\, dx, \text{ as before.}$$

Both the above methods are based directly on first principles. We may also obtain the differential by a purely calculus method, thus,

$$d\tan x = d\left(\frac{\sin x}{\cos x}\right)$$

$$=\frac{\cos x\, d\sin x - \sin x\, d\cos x}{\cos^2 x}, \text{ by Art. 27,}$$

$$=\frac{\cos^2 x\, dx + \sin^2 x\, dx}{\cos^2 x}$$

$$=\frac{dx}{\cos^2 x} = \sec^2 x\, dx.$$

39. *To find the differential of* $\cot x$.

The most instructive way will be by means of the identity

$$\cot x = \tan\left(\frac{\pi}{2} - x\right);$$

$$\therefore\ d\cot x = d\tan\left(\frac{\pi}{2} - x\right)$$

$$=\sec^2\left(\frac{\pi}{2} - x\right) d\left(\frac{\pi}{2} - x\right)$$

which reduces to $-\operatorname{cosec}^2 x\, dx$, since $d\left(\frac{\pi}{2} - x\right) = -dx$.

i.e. $\mathbf{d\cot x = -cosec^2 x\, dx}$ and $\mathbf{D\cot x = -cosec^2 x.}$

The student may also practise finding the differential by means of the identity $\cot x = \frac{\cos x}{\sin x}$, and using the quotient formula.

40. *To find* $d(\sec x)$.

The easiest way is as follows:—

$$
\begin{aligned}
d\sec x &= d(\cos x)^{-1} \\
&= -(\cos x)^{-2}\, d\cos x \\
&= -\frac{1}{\cos^2 x}\cdot(-\sin x\, dx) = \frac{\sin x}{\cos^2 x}dx,
\end{aligned}
$$

i.e. $\mathbf{d\sec x = \sec x \tan x\, dx.}$

41. *To find* $d(\operatorname{cosec} x)$.

$$
\begin{aligned}
d(\operatorname{cosec} x) &= d\sec\left(\frac{\pi}{2}-x\right) \\
&= \sec\left(\frac{\pi}{2}-x\right)\tan\left(\frac{\pi}{2}-x\right)d\left(\frac{\pi}{2}-x\right),
\end{aligned}
$$

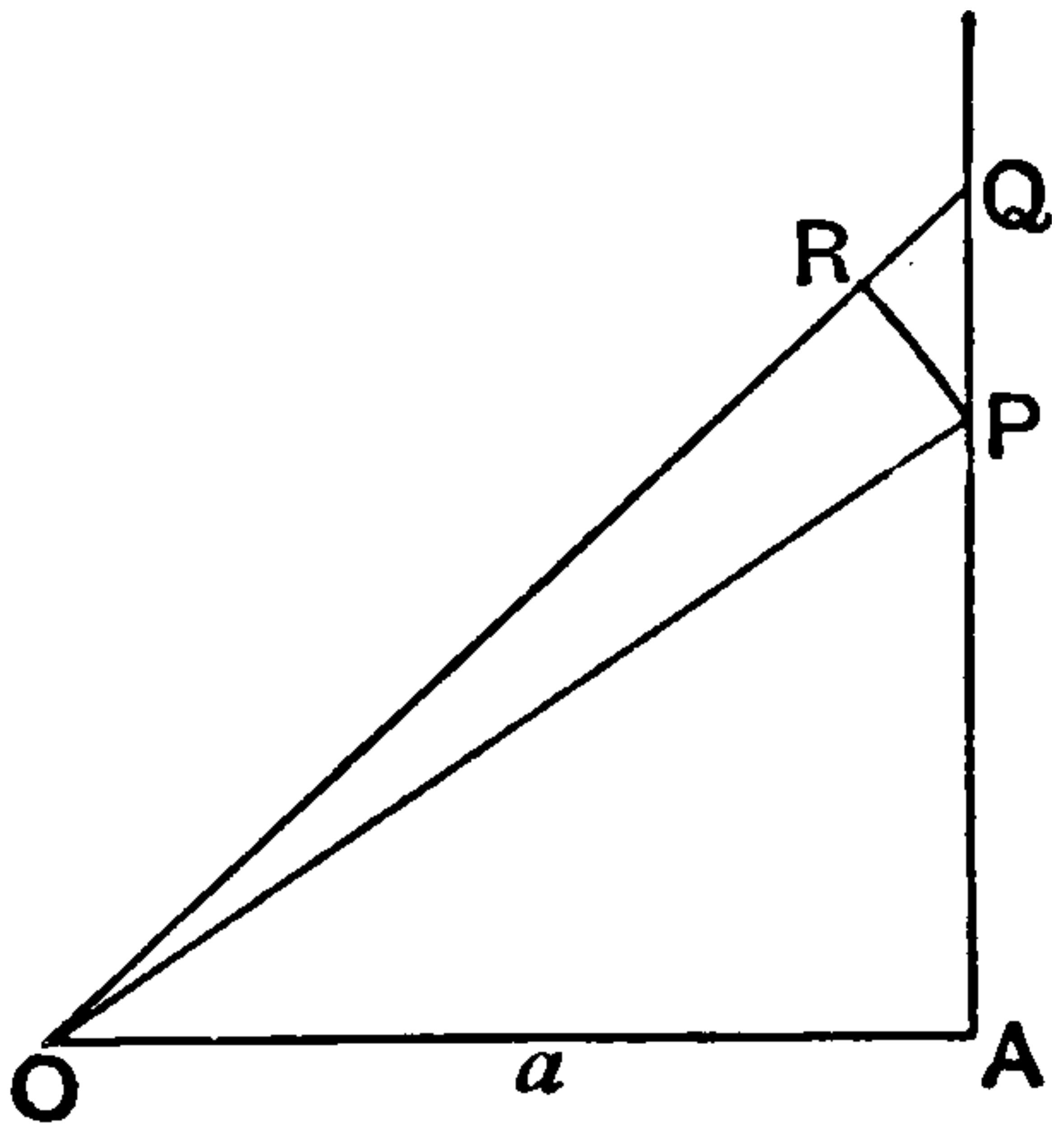
i.e. $\mathbf{d\operatorname{cosec} x = -\operatorname{cosec} x \cot x\, dx.}$

42. *Geometrical method of finding* $d\tan x$ *and* $d\sec x$ *in terms of* dx.

Let $\quad\mathrm{POA} = x,$

$\mathrm{QOA} = x + dx.$

Draw PR perpendicular to both OP and OQ (Art. 32), cutting OQ in R.



Then $\quad dx = \dfrac{\mathrm{RP}}{\mathrm{OP}}$, if dx is in circular measure.

Also
$$d\tan x=\frac{\mathrm{QA}}{a}-\frac{\mathrm{PA}}{a}=\frac{\mathrm{QP}}{a}=\frac{\mathrm{QP}}{\mathrm{RP}}\cdot\frac{\mathrm{RP}}{a}$$
$$=\frac{\mathrm{QP}}{\mathrm{RP}}\cdot\frac{\mathrm{RP}}{\mathrm{OP}}\cdot\frac{\mathrm{OP}}{a}$$
$$=\sec \mathrm{QPR}.dx.\sec x.$$

But QPR is ultimately equal to x, being the complement of OPA, since OPR is a right angle;

$$\therefore\ d\tan x=\sec^2 x\,dx.$$

Similarly,
$$d\sec x=\frac{\mathrm{QR}}{a},\ \text{since}\ \mathrm{OP}=\mathrm{OR},$$
$$=\frac{\mathrm{QR}}{\mathrm{RP}}\cdot\frac{\mathrm{RP}}{\mathrm{OP}}\cdot\frac{\mathrm{OP}}{a}$$
$$=\tan x.dx.\sec x$$
$$=\sec x\tan x\,dx.$$

43. The six circular function formulae are therefore

$$d\sin x=\cos x\,dx$$
$$d\cos x=-\sin x\,dx,$$
$$d\tan x=\sec^2 x\,dx$$
$$d\cot x=-\operatorname{cosec}^2 x\,dx,$$
$$d\sec x=\sec x\tan x\,dx$$
$$d\operatorname{cosec} x=-\operatorname{cosec} x\cot x\,dx.$$

The above equations fall naturally into pairs, each of the functions $\sin x$, $\tan x$, $\sec x$, which increase with x (in the first quadrant), being associated with one of the complementary functions $\cos x$, $\cot x$, $\operatorname{cosec} x$ which decrease when x increases (in the first quadrant), and whose differentials have accordingly negative signs.

This association in pairs will materially assist the student in remembering the formulae.

In addition to this complementary pairing of equations, there is also a natural association between the pair of functions which occur in the same equation; it will be found much better to remember $d\tan x$ as $=\sec^2 x\,dx$ than as $\frac{dx}{\cos^2 x}$, and similarly $d\sec x = \sec x \tan x\,dx$ rather than $\frac{\sin x\,dx}{\cos^2 x}$, since $\sec x$ and $\tan x$ form a natural pair of connected quantities, just as $\sin x$ and $\cos x$ form a natural pair. Similarly $\operatorname{cosec} x$ and $\cot x$ form a natural pair of associated functions.

This natural pairing is well shown in the three important formulae:

$$\sin^2 x + \cos^2 x = 1,$$

$$\sec^2 x - \tan^2 x = 1,$$

$$\operatorname{cosec}^2 x - \cot^2 x = 1,$$

which are the three trigonometrical forms of Euc. I. 47.

The student may, in fact, derive $d\sec x$ from $d\tan x$ by differentiating the second of these three identities, and he may derive $d\operatorname{cosec} x$ from $d\cot x$ by means of the third, just as we have already derived $d\cos x$ from $d\sin x$ by means of the first.

44. Inverse circular functions. The differentials of the inverse functions $\sin^{-1}x$, $\cos^{-1}x$, etc., can be readily deduced from those of the direct functions. For example:

45. *To find the differential of* $\sin^{-1}x$.

Let $\theta = \sin^{-1}x$

$$\therefore \sin\theta = x$$

$$\therefore \cos\theta\,d\theta = dx$$

$$\therefore d\theta = \frac{dx}{\cos\theta} = \frac{dx}{\sqrt{(1-x^2)}}.$$

i.e. $$\mathbf{d\sin^{-1}x = \frac{dx}{\sqrt{(1-x^2)}}}.$$

The latter form, with the surd, is ambiguous. The sign to be attached to the surd must be + or −, according as $\cos\theta$ is positive or negative. In the case of acute angles it is of course positive.

46. *To find the differential of* $\cos^{-1}x$.

This can be found in the same way by putting $\theta=\cos^{-1}x$, whence finally
$$d\theta=-\frac{dx}{\sin\theta}=-\frac{dx}{\sqrt{(1-x^2)}};$$
$$\text{i.e. } \mathbf{d\cos^{-1}x=-\frac{dx}{\sqrt{(1-x^2)}}}.$$

The ambiguity in the surd must be disposed of in a similar way to that of the last case, only in this instance the sign to be attached to the surd is that of $\sin\theta$.

The following method is applicable in the case of angles in the first quadrant. The angle whose cosine is x is the complement of the angle whose sine is x,
$$\text{i.e. } \cos^{-1}x+\sin^{-1}x=\frac{\pi}{2}$$
$$\therefore\ d(\cos^{-1}x)+d(\sin^{-1}x)=0$$
i.e. the differential of $\cos^{-1}x$ is numerically equal to that of $\sin^{-1}x$, but of opposite sign.

47. The student should draw the graph of $\sin^{-1}x$, *i e.* the curve $y=\sin^{-1}x$, plotting the values of y in radians, *i.e.* in circular measure. He will see that y is a multi-valued function of x, and that for each value of x the gradients at the different points of the graph corresponding to the different values of y are alternately + and −, being numerically equal to $\frac{1}{\sqrt{(1-x^2)}}$. If y_0 is the smallest positive value of y corresponding to any value of x, the other values of y for that abscissa are $\pi-y_0$, $2\pi+y_0$, $3\pi-y_0$, etc., all being included in the expression $n\pi+(-1)^n y_0$ where n is any integer, positive or negative. (The use of the factor $(-1)^n$ is a very convenient mode of indicating that a + sign is required when n is even, and a − sign when n is odd.)

It would be instructive to also draw the graph of $\cos^{-1}x$, which is merely the $\sin^{-1}x$ curve brought down through a distance $=\frac{\pi}{2}$ on the vertical scale; or, which comes to the same thing, it is actually the same graph as $\sin^{-1}x$, but with a new origin and x-axis at a height $=\frac{\pi}{2}$ *above* the old one.

48. *To find the differential of* $\tan^{-1}x$, *i.e.* to find $d\theta$ when $\tan\theta = x$.

We have $\sec^2\theta \, . \, d\theta = dx$

$$\therefore \; d\theta = \frac{dx}{\sec^2\theta} = \frac{dx}{1+x^2}, \text{ since } \sec^2\theta = 1 + \tan^2\theta.$$

i.e. $$\mathbf{d \tan^{-1} x = \frac{dx}{1+x^2}}.$$

49. *To find the differential of* $\cot^{-1}x$.

Proceeding exactly in the same way, we find

$$\mathbf{d \cot^{-1} x = -\frac{dx}{1+x^2}}.$$

There is no ambiguity of sign in these expressions.

50. *To find the differentials of* $\sec^{-1}x$ *and* $\operatorname{cosec}^{-1}x$.

If $\theta = \sec^{-1}x$, $\sec\theta = x$;

$$\therefore \; d\theta = \frac{dx}{\sec\theta \tan\theta},$$

i.e. $$\mathbf{d \sec^{-1} x = \frac{dx}{x\sqrt{(x^2-1)}}}.$$

Similarly,

$$\mathbf{d \operatorname{cosec}^{-1} x = -\frac{dx}{x\sqrt{(x^2-1)}}}.$$

In these expressions there is an ambiguity of sign, the sign to be given to the surd being the same as that of $\tan\theta$.

51. Recapitulation.

$$\left.\begin{aligned} d\sin\theta &= \cos\theta \, . \, d\theta \\ d\cos\theta &= -\sin\theta \, . \, d\theta \end{aligned}\right\}$$

$$\left.\begin{aligned} d\tan\theta &= \sec^2\theta \, . \, d\theta \\ d\cot\theta &= -\operatorname{cosec}^2\theta \, . \, d\theta \end{aligned}\right\}$$

$$\left.\begin{aligned} d\sec\theta &= \sec\theta\tan\theta\,d\theta \\ d\,\mathrm{cosec}\,\theta &= -\mathrm{cosec}\,\theta\cot\theta\,d\theta \end{aligned}\right\}$$

$$\left.\begin{aligned} &d\sin^{-1}x \\ &= -d\cos^{-1}x \end{aligned}\right\} = \frac{dx}{\sqrt{(1-x^2)}}.$$

$$\left.\begin{aligned} &d\tan^{-1}x \\ &= -d\cot^{-1}x \end{aligned}\right\} = \frac{dx}{1+x^2}.$$

$$\left.\begin{aligned} &d\sec^{-1}x \\ &= -d\,\mathrm{cosec}^{-1}x \end{aligned}\right\} = \frac{dx}{x\sqrt{(x^2-1)}}.$$

EXAMPLES.

Find $\frac{dy}{dx}$ in the five following examples:

1. $y=\cos^2x+\sin^2x$.

2. $y=\cos^2x-\sin^2x$.

3. $y=\tan 3x-\cot 4x$.

4. $y=\frac{\sin x-\sin 3x}{4\cos^3x}$. [NOTE. $4\cos^3x=3\cos x+\cos 3x$.]

5. $y=\sec x+\tan x$.

6. Obtain another identity by differentiating both sides of the identity
$$\sin 2x=2\sin x\cos x.$$

7. Differentiate both sides of the identity
$$\cos 3x=4\cos^3x-3\cos x.$$

8. Differentiate $\sec^4x-4\sec^3x+2\sec^2x$.

9. Differentiate $(3\sec^4x+4\sec^2x+8)\tan x$.

10. Differentiate $\sin^{-1}(2x^2)$.

11. Differentiate $x\sin^{-1}x+\sqrt{(1-x^2)}$.

12. If $f(x)=(x^2+1)\tan^{-1}x-x$, find $f'(x)$.

13. If $f(x)=\tan^{-1}\frac{x}{\sqrt{(1-x^2)}}$, find $f'(x)$.

14. If $\tan\theta=\sqrt{(x^2+1)}-x$, find $\frac{d\theta}{dx}$.

15. If $(1+x^2)\cos\theta=1-x^2$, find $\frac{d\theta}{dx}$.

16. Draw the curve $y=\frac{1}{2}x\sin\frac{\pi x}{2}$, showing that it touches the lines $y=\pm\frac{1}{2}x$ (alternately) when $x=1, 3, 5\ldots$, and finding where and at what gradients it crosses the zero line ($y=0$), and determining also the points where the gradient is zero.

17. Draw the curve $y=\frac{2}{x}\sin\frac{\pi x}{2}$, showing that it oscillates between the curves $y=\pm\frac{2}{x}$ (which should also be drawn). Find where and at what slopes it crosses the zero line, and at what points the gradient is zero.

18. Draw similarly the curves $y=\frac{1}{2}x\sin\pi x$, and $y=\frac{2}{x}\sin\pi x$.

19. Draw the curve $y=\frac{2}{x}\cos\frac{\pi x}{2}$.

20. Draw the curve $y=\frac{1}{2}x\cos\frac{\pi x}{2}$.

52. Expansion of a function in powers of x.

We are now in a position to determine two important series giving the values of $\sin x$ and $\cos x$ numerically in terms of the circular measure of the angle x.

Before proceeding to these series, it may be well to illustrate what is meant by expanding a function of x in a series of powers of x. We know that

$$(1+x)^5=1+5x+10x^2+10x^3+5x^4+x^5.$$

The right-hand expression is a series consisting of positive integral powers of x, the first term being of zero power, and the last of the 5th power. This series, then, is said to be the expansion of $(1+x)^5$ in positive integral powers of x.

We could have determined this series by the help of the calculus in the following way.

Any series consisting of positive integral powers of x must be of the form

$$a_0+a_1x+a_2x^2+a_3x^3+\dots,$$

or, if we prefer so to write it, in the form

$$a_0+a_1x+a_2\frac{x^2}{\underline{|2}}+a_3\frac{x^3}{\underline{|3}}+\dots,$$

where, in either case, a_0, a_1, $a_2\dots$ are numerical coefficients.

For the method we are now going to illustrate the second form of the series is usually rather more convenient. What we shall do is to first assume that the given function $(1+x)^5$ is equal to this series. Then, since differentiating both sides of an identity leads to another identity, we shall, b differentiating, obtain a second identical equation. Repeatin the same operation as many times as we want, we obtain whole set of equations from which we may discover the value of the coefficients a_0, a_1.... The whole work is very simple a soon as the student has got into the spirit of it. It is a follows:

Let $$(1+x)^5 = a_0 + a_1x + a_2\frac{x^2}{\underline{|2}} + a_3\frac{x^3}{\underline{|3}} + \dots, \quad \dots\dots\dots\dots(1$$

Then, differentiating, we have

$$5(1+x)^4 d(1+x) = a_1 dx + a_2 . x\,dx + a_3\frac{x^2}{\underline{|2}}dx + \dots,$$

and $$d(1+x) = dx,$$

∴ dividing by dx we have

$$5(1+x)^4 = a_1 + a_2x + a_3\frac{x^2}{\underline{|2}} + a_4\frac{x^3}{\underline{|3}} + \dots. \quad \dots\dots\dots\dots(2)$$

Now differentiate both sides of this identity, and divide by dx,

$$5 . 4(1+x)^3 = a_2 + a_3x + a_4\frac{x^2}{\underline{|2}} + \dots. \quad \dots\dots\dots\dots(3)$$

Repeating the operation again and again, we have

$$5 . 4 . 3(1+x)^2 = a_3 + a_4x + a_5\frac{x^2}{\underline{|2}} + \dots. \quad \dots\dots\dots\dots(4)$$

$$5 . 4 . 3 . 2(1+x) = a_4 + a_5x + \dots. \quad \dots\dots\dots\dots(5)$$

$$5 . 4 . 3 . 2 . 1 = a_5 + a_6x + \dots. \quad \dots\dots\dots\dots(6)$$

$$0 = a_6 + a_7x + \dots. \quad \dots\dots\dots\dots(7)$$

Now since we are assuming that the values of the coefficients a_0, a_1, ... are to be such as will make all the above equations *identities*, it follows that the equations will hold for all values of x. Therefore, put $x = 0$ in each of them, and we find

from (1), $a_0 = 1$,
(2), $a_1 = 5$,
(3), $a_2 = 5 . 4$,
etc.;

$\therefore$ finally, $(1+x)^5 = 1 + 5x + \frac{5 \cdot 4x^2}{\lfloor 2} + \frac{5 \cdot 4 \cdot 3x^3}{\lfloor 3} + \ldots,$

which reduces to $1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5$.

The student should also go through the work, assuming that

$$(1+x)^5 = a_0 + a_1x + a_2x^2 + \ldots .$$

He will obtain the same result, with perhaps rather less labour, but the method above indicated shows the law of formation of the coefficients rather more clearly. He should also expand $(1+x)^n$, using either form of series. The expansion found will be a terminating one (ending at x^n) if n is a positive integer, but for other values of n it will consist of an infinite series, and is then valid only if x lies between ±1.

53. Expansion of $\sin x$ and $\cos x$ in powers of x.

Now, let us apply the same method to expand $\sin x$ in positive integral powers of x.

Let $$\sin x = a_0 + a_1x + a_2\frac{x^2}{\lfloor 2} + a_3\frac{x^3}{\lfloor 3} + \ldots \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Differentiate time after time as we did above, and we get

$$\cos x = a_1 + a_2x + a_3\frac{x^2}{\lfloor 2} + a_4\frac{x^3}{\lfloor 3} + \ldots, \ldots\ldots\ldots\ldots\ldots (2)$$

$$-\sin x = a_2 + a_3x + \ldots, \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

$$-\cos x = a_3 + a_4x + \ldots, \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

$$\sin x = a_4 + \ldots, \ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

and so on, the left-hand side repeating itself in the same order as before, viz., $\sin x$, $\cos x$, $-\sin x$, $-\cos x$, ...

Now put $x = 0$ in each of these identities, and we get

$$0 = a_0 = a_2 = a_4 = a_6 = \ldots,$$

$$1 = a_1 = -a_3 = a_5 = -a_7 = \ldots$$

$\therefore$ finally, $\sin x = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \ldots$ ad inf.

We can expand $\cos x$ in the same way; but as a matter of fact we have already obtained $\cos x$, for it is given in (2), and is

$$\cos x = 1 - \frac{x^2}{\lfloor\underline{2}} + \frac{x^4}{\lfloor\underline{4}} - \dots .$$

54. We could have determined these series by two differentiations only. For equation (3) tells us that

$$\sin x = -a_2 - a_3 x - a_4 \frac{x^2}{\lfloor\underline{2}} - \dots .$$

Hence this series must be identical with the series

$$a_0 + a_1 x + a_2 \frac{x^2}{\lfloor\underline{2}} + \dots ;$$

therefore, by equating coefficients, we find that

$$a_2 = -a_0,\ a_3 = -a_1,\ a_4 = -a_2, \text{ etc.};$$

further, by putting $x = 0$ in (1) and (2) we find $a_0 = 0$, and $a_1 = 1$;

$$\therefore\ a_2 = 0,\ a_3 = -1,\ a_4 = 0, \text{ etc.}$$

This determines all the constants, and gives us the series for $\sin x$ obtained above.

55. These two series, viz.:

$$\sin x = x - \frac{x^3}{\lfloor\underline{3}} + \frac{x^5}{\lfloor\underline{5}} - \dots,$$

$$\cos x = 1 - \frac{x^2}{\lfloor\underline{2}} + \frac{x^4}{\lfloor\underline{4}} - \dots,$$

are very important. They are easily remembered: $\sin x$ contains all the odd powers of x, each divided by the corresponding factorial, and affected with alternate signs; $\cos x$ contains all the even powers (starting with zero power), each divided by the corresponding factorial, and affected with alternate signs. Also if the student differentiates the series for $\sin x$ he will obtain the series for $\cos x$, and if he differentiates the series for $\cos x$ he will obtain *minus* the series for $\sin x$, as of course he should.

$\cos x$ is called an *even* function of x, as its expansion contains only even powers of x, which is in accordance with the fact that $\cos(-x) = +\cos x$.

$\sin x$ is called an *odd* function of x, since all the powers of x are odd in its expansion. This agrees with the fact that $\sin(-x) = -\sin x$, from which it is evident no even powers of x can occur in the expansion, for even powers could not change sign when $-x$ is put for x.

56. We could use the above method to expand $\tan x$ and $\sec x$, but we should only succeed in obtaining the successive terms by a good deal of labour because their successive differential coefficients become very troublesome. We can of course help ourselves somewhat by noticing that $\tan x$ is an odd function of x, since $\tan(-x) = -\tan x$; and by noticing that $\sec x$ is an even function of x. We shall return to these expansions in a subsequent chapter.

We could obtain a few terms of $\tan x$ by dividing $\sin x$ by $\cos x$, and a few terms of $\sec x$ by dividing 1 by $\cos x$ (using, of course, the series just found for these functions). This is left as an exercise for the student.

EXAMPLES.

1. Find 3 terms of the series for $\tan x$ by dividing

$$x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} \text{ by } 1 - \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}}.$$

2. Find 3 terms of the series for $\sec x$ by dividing 1 by $1 - \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}}$.

3. Find two terms of the series for $\tan x$ by assuming it equal to $a_1 x + a_3 \frac{x^3}{\underline{|3}} + \ldots$, and differentiating three times.

4. Expand $\cos 2x$ in powers of x. [Merely substitute $2x$ for x in the series for $\cos x$.]

5. Expand $\cos^2 x$ and $\sin^2 x$ in powers of x. [Make use of the facts that $\cos^2 x - \sin^2 x = \cos 2x$, and $\cos^2 x + \sin^2 x = 1$.]

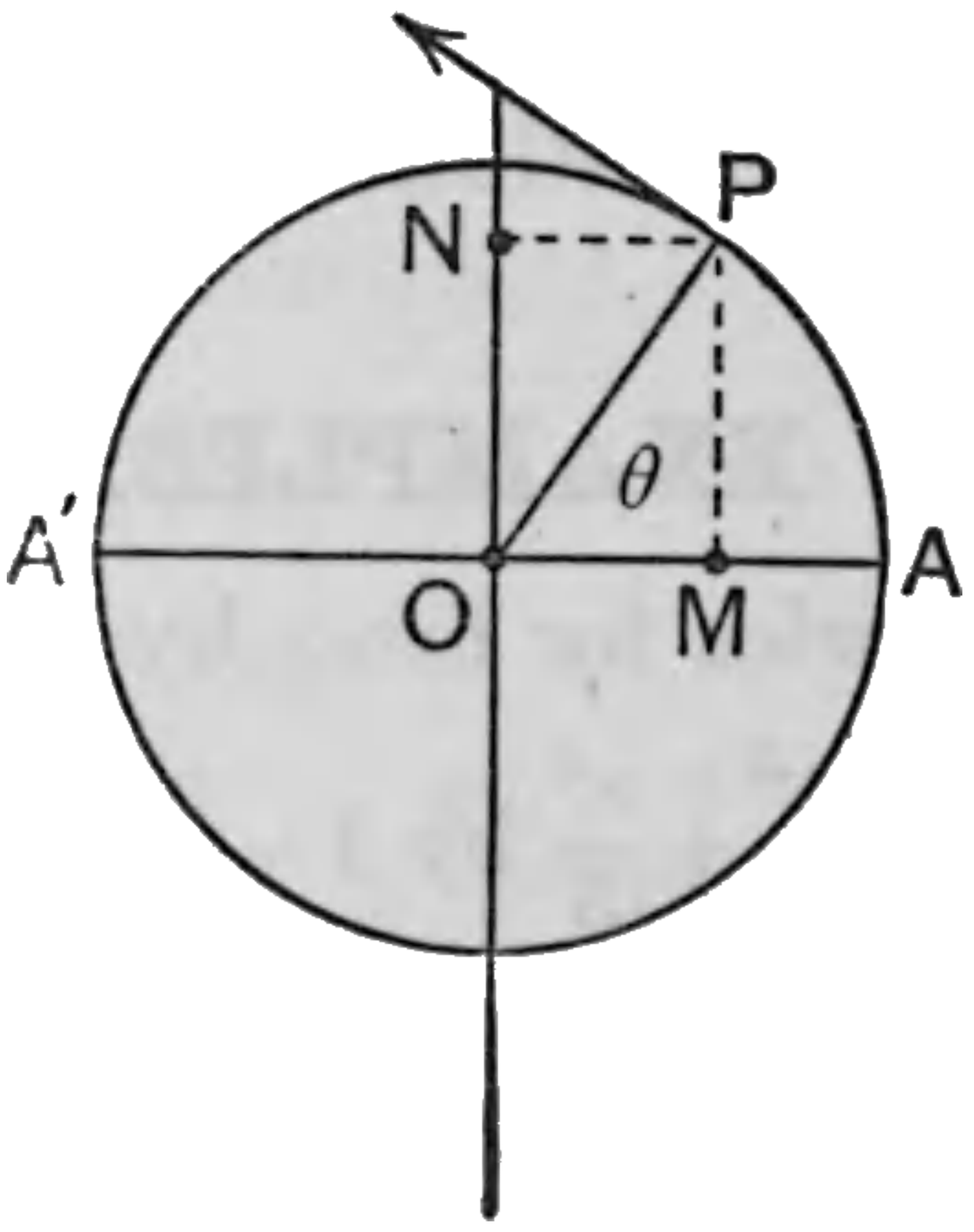
6. Draw the graph of $x - \frac{x^3}{\underline{|3}}$, and compare it with the graph of $\sin x$. Make a closer approximation by adding the term $\frac{x^5}{\underline{|5}}$.

7. Draw the graph of $1-\frac{x^2}{\underline{|2}}$, and compare it with the graph of $\cos x$. Make a closer approximation by adding the term $\frac{x^4}{\underline{|4}}$, and then a closer approximation still by subtracting $\frac{x^6}{\underline{|6}}$.

57. Uniform circular motion. One of the simplest illustrations of the values of the differential coefficients of $\sin\theta$ and $\cos\theta$ will be found in connection with the motion of a particle travelling with uniform velocity along the circumference of a circle.

If the radius is a, and the particle P has moved in time t from an initial position A through an arc $a\theta$, θ being the circular measure of the angle subtended by the arc AP at the centre O of the circle, the velocity of the particle is $v=\frac{a\theta}{t}$.

The position of the particle can be indicated by measuring the base OM and the height MP of the right-angled triangle OMP.



Denote OM by x, and MP by y; then

$$x=a\cos\theta,\quad y=a\sin\theta.$$

The component velocities of P in the x and y directions respectively are

$$\frac{dx}{dt} \text{ and } \frac{dy}{dt}.$$

Since $x=a\cos\theta$, it follows that

$$\frac{dx}{dt}=-a\sin\theta\frac{d\theta}{dt};$$

and since $y=a\sin\theta$, it follows that

$$\frac{dy}{dt}=a\cos\theta\frac{d\theta}{dt}.$$

We can obtain these component velocities in another way, for it is evident from the figure that the component velocities of P are $-v\sin\theta$ and $+v\cos\theta$ respectively, and since v is constant, and equal to $\frac{a\theta}{t}$, *i.e.* $vt=a\theta$, we have, on differentiating, $vdt=ad\theta$, whence $v=a\frac{d\theta}{dt}$.

Hence, $$-v\sin\theta=-a\sin\theta\frac{d\theta}{dt},$$

and $$v\cos\theta=a\cos\theta\frac{d\theta}{dt},$$

agreeing with the component velocities obtained by the first method, viz., by direct differentiation of the values of x and y.

58. If the uniform circular motion of a particle is looked at edgeways from a long way off, that is to say, if it is orthogonally projected on to a line in the plane of the circle, as for instance on to a diameter, the projected motion is a simple oscillation, the simplest kind of oscillation possible, the kind of motion possessed by a point on a musical instrument sounding a pure tone, as for instance a bead on a tuning fork; and it is therefore called a pure or simple harmonic motion, of which the properties can easily be laid down by considering the circular motion above.

59. Simple harmonic motion. If a particle P moves with uniform speed round a circle, and a point M moves along a diameter AA′ of the circle with such variable speed that the line PM is always perpendicular to AA′, M is said to have a simple harmonic motion. The radius, OA, of the circle is called the *amplitude* of the simple harmonic motion. Evidently the motion of M is a to-and-fro motion between A and A′, the velocity being zero at A and A′ and a maximum at the middle point O, being then equal to the velocity of P. At any intermediate point M, such that OM $=x=a\cos\theta$, the velocity of $M=\frac{dx}{dt}=-a\sin\theta\frac{d\theta}{dt}$, one complete oscillation being described while θ changes from 0 to 2π; *i.e.* in the time during which P describes a whole revolution. This time, T, is called the period of oscillation, and is evidently equal to the length of the circular path described by P, divided by its velocity v; *i.e.* $=\frac{2\pi a}{v}$.

60. Combinations of simple harmonic motions of equal periods. In the figure, when P moves uniformly round the circle, the points M and N move along their respective diameters with simple harmonic motion, the velocity of M being equal to the horizontal component of the velocity of P, and the velocity of N being equal to the vertical component of the velocity of P. Moreover, when M is at A or A′, N is at O, and when N is at B or B′, M is at O, so that one is a quarter oscillation ahead of the other, while the amplitudes and periods of their oscillations are equal.

This gives us the following theorem:

Uniform motion round the circumference of a circle can be considered as the resultant of two simple harmonic motions of equal period and amplitude along two intersecting lines at right angles to each other, so arranged that one is a quarter oscillation ahead of the other.

If the periods are equal, but either the amplitudes are unequal, or one of the points M, N is not a quarter oscillation ahead of the other, the resultant motion will be not along a circle but along an ellipse. A point moving in such manner is said to have elliptic harmonic motion.

EXAMPLES.

1. Draw the curves

(1) $x=a\cos\theta,\quad y=a\cos\theta,$
(2) $x=a\cos\theta,\quad y=a\cos(\theta+18°),$
(3) $x=a\cos\theta,\quad y=a\cos(\theta+36°),$
(4) $x=a\cos\theta,\quad y=a\cos(\theta+54°),$
(5) $x=a\cos\theta,\quad y=a\cos(\theta+72°),$
(6) $x=a\cos\theta,\quad y=a\cos(\theta+90°).$

[NOTE.—Take a some convenient length, say 5 inches if the paper is large enough (*i.e.* 10 inches wide at least), and calculate the values of x and y for values of θ from 0° to 360° at intervals of 18°. Then draw a network of horizontal and vertical lines through the points corresponding to the calculated values of x and y. It will be found that the same network serves for all the above curves.]

2. Show that the tangent to any of the above curves at the point where $x=a\cos\theta$ is parallel to the line drawn from the origin to the point on the same curve whose abscissa is $a\cos(\theta+90°)$. Test the accuracy of this result by trial on the diagram.

3. Draw the curves

$$x=5\cos\theta, \quad y=3\cos(\theta+\alpha),$$

for the following values of α, viz. 0°, 18°, 36°, 54°, 72°, 90°.

[NOTE.—Form a similar network to that in No. 1, except that the distances of the horizontal lines from the origin will be 3, $3\cos 18°$, $3\cos 36°$, ..., while the distances of the vertical lines from the origin will be 5, $5\cos 18°$, $5\cos 36°$, ...]

4. Show that each portion of the networks indicated in connection with the above examples is described in the same time, viz., $\frac{1}{20}T$; where T is the period of the complete motion.

CHAPTER IV.

THE EXPONENTIAL AND LOGARITHMIC FUNCTIONS a^x AND $\log x$.

61. The function first to be considered in this chapter is a^x. The student should draw its graph for a few simple values of a [see Ex. 1, p. 56]; he will then see that it crosses the axis of y always at the point (0, 1), but at different gradients, according to the value of a chosen, the slope at this point being greater or less than 45°, according to the value chosen for a.

We shall now show that the gradient at any other point is intimately and simply connected with the gradient at (0, 1). Let $\tan\phi$ be the gradient at any point (x, y) of the curve, and $\tan\phi_0$ the gradient at (0, 1); we shall find that $\tan\phi = y\tan\phi_0$.

For, let
$$y = a^x$$
then
$$y + dy = a^{x+dx}$$
$$\therefore\ dy = a^{x+dx} - a^x = a^x(a^{dx} - 1)$$
$$\therefore\ \frac{dy}{dx} = a^x \cdot \frac{a^{dx} - 1}{dx}.$$

Hence, remembering that $\frac{dy}{dx}$ is $\tan\phi$, and becomes $\tan\phi_0$ when $x = 0$, *i.e.* when $a^x = 1$, we have

$$\tan\phi = a^x \tan\phi_0 = y\tan\phi_0, \text{ as stated above.}$$

Now we have seen that the value of $\tan\phi_0$ depends on the value of a chosen; therefore, in the first instance, let us take a value of a (at present unknown) which will make $\tan\phi_0 = 1$.

Denote this value of a by the symbol e, and proceed to find its value. It is evidently an important number, for the differential of e^x will be simpler than that of any other exponential function.

This method of considering the variation of a^x by help of its graph was suggested by Prof. E. J. Nansen.

The condition that $\tan\phi_0 = 1$ when $a = e$, gives us the equations
$$\mathbf{de^x = e^x dx}, \quad \text{and} \quad \mathbf{De^x = e^x}.$$

Thus the differential coefficient of e^x is equal to e^x. The value of e can be determined without much difficulty from this property of its differential coefficient.

Expand e^x in powers of x by putting

$$e^x = a_0 + a_1 x + a_2 \frac{x^2}{\underline{|2}} + a_3 \frac{x^3}{\underline{|3}} + \dots, \quad \dots\dots\dots\dots\dots (1)$$

and differentiating both sides of the assumed identity just as was done in the cases of $\sin x$ and $\cos x$, thus obtaining

$$\mathrm{D}e^x = a_1 + a_2 x + a_3 \frac{x^2}{\underline{|2}} + \dots. \quad \dots\dots\dots\dots\dots\dots (2)$$

But $\mathrm{D}e^x = e^x$, therefore the series (2) is identical* with the series (1);

$$\therefore \; a_1 = a_0, \; a_2 = a_1, \; a_3 = a_2, \text{ etc.}$$

By putting $x = 0$ in (1) we find that $a_0 = e^0 = 1$,

$$\therefore \; a_1 = 1, \; a_2 = 1, \; a_3 = 1, \dots;$$

∴ finally

$$\mathbf{e^x = 1 + x + \frac{x^2}{1 . 2} + \frac{x^3}{1 . 2 . 3} + \dots.}$$

This series is very important. It is known as the exponential series, e^x being called the exponential function.

If we put $x = 1$, in this identity, we find

$$\mathbf{e = 1 + 1 + \frac{1}{1 . 2} + \frac{1}{1 . 2 . 3} + \dots.}$$

This gives the required value of e. On working it out we find $e = 2{\cdot}718281828\dots$.

It can be calculated to as many figures as the patience and accuracy of the computer permit, but the important part of its value, not to be ever forgotten, is 2·7. After that the next eight figures, which are 1828 repeated twice (but then followed by other figures), may also be easily remembered.

With this value of e, then, we know that

$$de^x = e^x dx.$$

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* This is true only if the infinitely distant terms of the series thus found are infinitely small, which on investigation is seen to be the case. A more thorough proof of the reliability of the series for all values of x is given in Chap. XI.; see Arts. 155, 156, 157.

Of course also, if u is any function of x, or any variable whatever,

$$de^u = e^u du.$$

Hence e is clearly an important number. We shall return to it directly, in connection with logarithms.

We can now find out the value of $\tan \phi_0$ in the expression for $d(a^x)$, viz., $a^x . \tan \phi_0 \, dx$.

For $a^x = e^{kx}$, where $e^k = a$, so that $k = \log_e a$.

Hence $$d(a^x) = d(e^{kx}) = e^{kx} d(kx),$$

i.e. $$d(a^x) = a^x k \, dx.$$

Therefore $$\tan \phi_0 = k = \log_e a,$$

i.e. $$a^{dx} - 1 = (\log_e a) \, dx.$$

We have proved, then, that

$$\mathbf{d(a^x) = ka^x dx} \quad \text{and} \quad \mathbf{D(a^x) = ka^x},$$

where $$\mathbf{k = \log_e a.}$$

It will be found that this constant k is of great importance in connection with the function a^x.

EXAMPLES.

1. Draw the curves $y = 2^x$, $y = 3^x$, and $y = 4^x$.

2. Draw the curve $y = e^x$, calculating the values of e^2, e^3, ... either by taking $e = 2{\cdot}7$ approximately, or by help of logarithms. Similarly for e^{-1}, e^{-2}, ... [Note that it lies between the two curves of No. 1.]

3. Show that the tangent at any point (x, y) on the curve $y = e^x$ cuts the axis of x at the point $(x - 1, 0)$, *i.e.* at a point situate unit distance to the left of the ordinate of the given point.

4. Draw the curves $y = e^{2x}$ and $y = e^{\frac{1}{2}x}$, and find in each case where the tangent at any point (x, y) on the curve cuts the axis of x.

5. Show that the sub-tangent of the curve $y = ae^{kx}$ is equal to $\frac{1}{k}$, whatever the length a may be. [The sub-tangent is the part of the axis of x intercepted between the tangent at any point and the ordinate of that point.]

6. Show that the curves $y = ae^{kx}$, and $y = be^{kx}$ are the same curve, with the origin so placed that in the first curve $y = a$ when $x = 0$, and, in the second curve, $y = b$ when $x = 0$.

7. Show that the curves $y = ae^{kx}$ and $y = ae^{-kx}$ are the same curves with their right and left portions interchanged.

8. Draw the curves $y=\frac{1}{2}(e^x+e^{-x})$ and $y=\frac{1}{2}(e^x-e^{-x})$. [Measure the required ordinates by help of the curve $y=e^x$.]

9. Draw the curve $y=10^{\frac{1}{10}x}$, and show that it is the same as $y=e^{kx}$ if $k=\frac{1}{10}\log_e 10=0{\cdot}23\ldots$.

10. Illustrate graphically the identity $e^x=1+x+\frac{x^2}{1\,.\,2}+\ldots$ by drawing the curves

(1) $y=1+x$,

(2) $y=1+x+\frac{x^2}{1\,.\,2}$,

(3) $y=1+x+\frac{x^2}{1\,.\,2}+\frac{x^3}{1\,.\,2\,.\,3}$,

between $x=\pm 6$. [Each curve can be built on its predecessor by adding to the ordinates the extra portions due to the next term.]

11. Find the series for a^x, expanded in powers of x, by assuming $a^x=a_0+a_1x+a_2\frac{x^2}{1\,.\,2}+\ldots$, and making use of the relation $\mathrm{D}(a^x)=ka^x$, where $k=\log_e a$.

Also deduce the series from the series for e^z, where $z=x\log_e a$.

12. Differentiate the series for e^x, and deduce the fact that $de^x=e^x dx$.

62. Consider again the equation $a^{dx}-1=\log_e a\,.\,dx$, *i.e.*,

$$\frac{a^{dx}-1}{dx}=\log_e a,$$

proved at the end of the last article.

It is usually written in the form,

the limiting value of $\frac{a^h-1}{h}$ is $\log_e a$, when $h=0$.

As a particular case,

the limiting value of $\frac{e^h-1}{h}$ is 1, when $h=0$.

This is also obvious from the series for e^h, viz.:

$$e^h=1+h+\frac{h^2}{1\,.\,2}+\ldots,$$

$$\therefore\ \frac{e^h-1}{h}=1+\frac{h}{1\,.\,2}+\ldots,$$

$$=1,\ \text{when } h=0.$$

These two relations are very important.

The following examples may help the student to realize their truth.

It may be instructive also to view them as approximate formulae when h is small but not quite zero, thus :

The student has already learnt that $e^0=1$, and that $a^0=1$. He now learns that $e^h=1+h$, where h is a small quantity, so small that its square and higher powers are negligible; and that $a^h=1+h\log_e a$ for similar small values of h, where a is any number whatever.

EXAMPLES.

1. As a verification of the fact that $\frac{a^x-1}{x}=\log_e a$ when $x=0$, plot the curve $y=\frac{10^x-1}{x}$, calculating the ordinates specially for the following values of x, ·3, ·2, ·1, $-$·1, $-$·2, $-$·3. The curve will be found to cross the line $x=0$ (the axis of y) at the point where $y=\log_e 10=2·3$ (approx.).

2. Calculate the values of $\frac{e^x-1}{x}$ for the same values of x, and draw the curve $y=\frac{e^x-1}{x}$, showing graphically that $y=1$ when $x=0$.

3. Putting $1+x$ for a in the fraction $\frac{a^h-1}{h}$, and expanding $(1+x)^h$ by the Binomial Theorem, prove that

$$\log_e(1+x)=x-\tfrac{1}{2}x^2+\tfrac{1}{3}x^3-\dots.$$

[NOTE.—This identity holds only in the range of values of x contained between $x=-1$ and $x=1$. This limitation of range will be discussed in Chapter XL]

4. Draw the curve $y=\log_e(1+x)$ between $x=-1$ and $x=2$, on a large scale. (The curve does not exist for negative values of x numerically greater than -1.)

5. Draw the curves (1) $y=x$;

(2) $y=x-\frac{1}{2}x^2$;

(3) $y=x-\frac{1}{2}x^2+\frac{1}{3}x^3$;

(4) $y=x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4$;

between the limits $x=\pm 2$, showing that as more terms are taken the curves lie closer to the curve $y=\log_e(1+x)$ so long as x is less than 1, but tend to diverge from this curve beyond that limit.

63. *To find the differential of* $\log_e x$.

The increase in $\log x$ for a given increase in x depends among other things on the base to which the logarithms are taken. The most important case in theoretical work is that in which the base is e, which is often called the natural base of logarithms, as theorems relating to the calculation of logarithms are simpler with that base than any other. It is also called the Napierian base as it, or rather its reciprocal, was used originally by Baron Napier, the discoverer of logarithms (*circa* A.D. 1624). The common base, 10, was suggested afterwards because of its greater advantage in numerical work. We are, then, taking first the case of $\log_e x$.

Let

$$y = \log_e x,$$

$$\therefore\ e^y = x,$$

$$\therefore\ e^y dy = dx,$$

$$\therefore\ dy = \frac{dx}{e^y} = \frac{dx}{x}.$$

i.e.

$$\mathbf{d(\log_e x) = \frac{dx}{x}}.$$

This is a very important theorem and should be remembered with absolute readiness.

We may prove it another way, viz.:—by differentiating $\frac{x^h - 1}{h}$, and then putting $h = 0$ (see last Article). Thus

$$d\frac{x^h - 1}{h} = \frac{1}{h}d(x^h - 1) = x^{h-1}dx,$$

which becomes $x^{-1}dx$ when $h = 0$.

The formula is true of course for *any* variable whatever,

e.g.

$$d(\log_e u) = \frac{du}{u},$$

where u is any function of x.

Thus, for instance,

$$d(\log_e x^2) = \frac{d(x^2)}{x^2} = \frac{2x\,dx}{x^2} = 2d(\log_e x).$$

Also

$$\text{D}\log_e(a + bx + cx^2) = \frac{b + 2cx}{a + bx + cx^2}.$$

64. The above theorem is often useful in the form

$$\mathbf{du = u\,d(\log_e u).}$$

In fact this formula gives the neatest way of arriving at the differential of a^x. For, let $u = a^x$, so that $\log_e u = x \log_e a$, $\therefore$ by the formula just written, we have

$$\begin{aligned} d\,a^x &= a^x d(x \log_e a), \\ &= a^x \log_e a \,.\, dx; \end{aligned}$$

and, in particular, the formula

$$d\,e^x = e^x dx$$

comes out at once,* since x is itself the log of e^x.

Examples, applying the above formula:

(1)
$$\begin{aligned} d(x^n) &= x^n d(n \log_e x) \\ &= x^n \,.\, \frac{n\,dx}{x} \\ &= n\,x^{n-1}dx, \text{ as proved long before.} \end{aligned}$$

(2) To take a harder case:

$$\begin{aligned} d(x^{\sin x}) &= x^{\sin x} d(\sin x \log_e x) \\ &= x^{\sin x}(\log_e x\, d\sin x + \sin x\, d\log_e x) \\ &= x^{\sin x}\left(\cos x \,.\, \log_e x + \frac{\sin x}{x}\right) dx. \end{aligned}$$

65. Methods of saving labour in differentiation. In many cases if we wish to differentiate a function consisting of a number of factors, it is a practical help to take logarithms first (to base e), and then differentiate. This is called logarithmic differentiation.

Thus, let
$$y = \frac{x\sqrt{1-x^2}}{\sqrt{1+x^2}};$$

$$\therefore \log y = \log x + \tfrac{1}{2}\log(1-x^2) - \tfrac{1}{2}\log(1+x^2);$$

$$\therefore \frac{dy}{y} = \frac{dx}{x} + \frac{1}{2}\frac{d(1-x^2)}{1-x^2} - \frac{1}{2}\frac{d(1+x^2)}{1+x^2};$$

---

* The student must guard against thinking that this method gives a *proof* of the formula for $d(e^x)$. It is merely interesting to see how neatly it comes out by this method. (See Ex. 25, p. 63.)

$$\therefore\ \frac{1}{y}\frac{dy}{dx}=\frac{1}{x}-\tfrac{1}{2}\cdot\frac{2x}{1-x^2}-\tfrac{1}{2}\cdot\frac{2x}{1+x^2}$$

$$=\frac{1}{x}-\frac{x}{1-x^2}-\frac{x}{1+x^2};$$

$$\therefore\ \frac{dy}{dx}=y\left(\frac{1}{x}-\frac{2x}{1-x^4}\right).$$

This finishes the differentiation, but, by inserting the value of y, some of the terms will cancel, as shown below.

$$\frac{dy}{dx}=\frac{x\sqrt{1-x^2}}{\sqrt{1+x^2}}\cdot\frac{1-2x^2-x^4}{x(1-x^4)}$$

$$=\frac{1-2x^2-x^4}{(1+x^2)^{\frac{3}{2}}(1-x^2)^{\frac{1}{2}}}.\qquad \textit{Ans.}$$

Another way of simplifying the process in this case would have been to square both sides instead of taking logs.

This would have given us

$$y^2=\frac{x^2-x^4}{1+x^2};$$

$$\therefore\ 2y\,dy=\frac{(1+x^2)(2x\,dx-4x^3dx)-(x^2-x^4).2x\,dx}{(1+x^2)^2};$$

$$\therefore\ y\frac{dy}{dx}=\frac{x(1-2x^2-x^4)}{(1+x^2)^2};$$

and then dividing by y, and replacing it by its value, the same result as before would have been obtained.

These examples illustrate methods of preparing a function for differentiation. Other methods in special cases may, with experience, suggest themselves to the student. If no obvious simplification suggests itself, the function must be carefully differentiated in the ordinary way, as it stands.

66. *To find the differential of* $\log_a x$.

Let $$y=\log_a x;$$

$$\therefore\ a^y=x;$$

$\therefore$ taking logs to base e,

$$y\log_e a=\log_e x;$$

$$\therefore\ \log_e a\,.\,dy=\frac{dx}{x};$$

$$\therefore\ dy=\frac{dx}{kx},$$

where $$k=\log_e a \text{ and } \frac{1}{k}=\log_a e.$$

In particular

$$d(\log_{10}x)=\mu\frac{dx}{x}$$

where $$\mu=\log_{10}e=\cdot 43429448\ldots.$$

The student should look out the common log of 2·7182818..., and verify to the first 6 or 7 figures that the above value of μ is correct.

67. The quantity μ is called the modulus of common logarithms, being the multiplier which converts napierian logarithms into common logs, the equation being

$$\log_{10}x=\mu\log_e x.$$

The proof of this may be left as an exercise to the student. He might also prove the following theorem, which is the fundamental theorem connected with the change of base of logarithms, viz.,

$$\mathbf{\log_a N=\frac{\log N}{\log a}},$$

where any base whatever may be used on the right-hand side, provided the numerator and denominator are taken to the same base. In words the theorem is

The log of a number to any base = the log of the number divided by the log of the base.

EXAMPLES.

[*For Recapitulation of formulae, see Art.* 76.]

Find $\frac{dy}{dx}$ in the following cases. All the logarithms are to base e.

1. $y=\log\sin x$. **2.** $y=\tan^{-1}x+\log\surd(1+x^2)$.

3. $y=\log(x+1)-\frac{2}{x+1}$. 4. $y=e^{x^2}$.

5. $y=x\log x$. 6. $y=\log(\sec x+\tan x)$.

7. $y=\log\frac{1+x+x^2}{1-x+x^2}$. 8. $y=a^{\sin x}$.

9. $\log y=x\sin x$. 10. $\log y=\frac{1}{x}\sin x$.

11. $y=x\,e^{-x^2}$. 12. $y=(\sin x)^{\cos x}$.

13. $y=\tan^2 x+\log(\cos^2 x)$. 14. $y=x^n\log x$.

15. $y=\log(\log x)$. 16. $y=\log(e^x+e^{-x})$.

17. From the series for e^x deduce the series for e^{ix} and e^{-ix} where i denotes $\sqrt{(-1)}$, *i.e.* $i^2=-1$.

18. Show that the series for $\frac{1}{2}(e^{ix}+e^{-ix})$ is equal to the series for $\cos x$.

19. Show similarly that $\frac{1}{2i}(e^{ix}-e^{-ix})=\sin x$.

20. Deduce that $\cos x\pm i\sin x=e^{\pm ix}$,
and that $(\cos x\pm i\sin x)^n=\cos nx\pm i\sin nx$.

21. Differentiate both sides of the identity

$$\log_e(1+x)=x-\tfrac{1}{2}x^2+\tfrac{1}{3}x^3-\ldots,$$

and prove by an elementary method that the resulting equation is an identity, provided x lies between the limits ± 1.

22. Expand $\log_e(1-x)$ and $\log_e\frac{1+x}{1-x}$ in powers of x.

23. By putting $x=\frac{1}{3}$ in the series for $\log_e\frac{1+x}{1-x}$, find a series for $\log_e 2$, and evaluate it to a few places of decimals.

24. Prove by help of Ex. 20 that $\log_e\frac{1+i\tan x}{1-i\tan x}=2ix$ and, by using the series for the left-hand side, deduce that

$$x=\tan x-\tfrac{1}{3}\tan^3 x+\tfrac{1}{5}\tan^5 x-\ldots.$$

25. If $y=e^x$, show that the equations $de^x=e^x dx$, and $dy=y\,d(\log_e y)$ are merely different forms of the same equation.

26. Differentiate $\frac{x^{1+h}-x}{h}$, and show that the differential coefficient becomes equal to that of $x\log_e x$ when $h=0$.

68. Compound interest property of exponential functions. The function a^x may be described as a function which grows at a rate proportional to itself, (if $a>1$).

For, if
$$y=a^x,$$
$$\frac{dy}{dx}=ka^x=ky,$$
where the numerical value of the constant k is
$$\log_e a\left(=\frac{\log_{10}a}{\log_{10}e}\right).$$
And the function a^{-x} is a function which diminishes at a rate proportional to itself, since, if
$$y=a^{-x},$$
$$\frac{dy}{dx}=-ky$$
where k has the same value as above. The connection of this constant with the rate of growth of $a^{\pm x}$ will be considered at greater length in the next article.

NOTE.—If a is less than 1, the rate of growth of a^x is negative, since k is then negative. In fact a^x is the same as b^{-x}, where b is equal to $\frac{1}{a}$, and is therefore greater than 1 if a is less.

We may therefore say that a^x is a function which grows or diminishes, always at a rate proportional to itself, according as a is greater or less than 1. In practice it is usually more convenient to take a greater than 1 and to make use of a^x or a^{-x} according as we are dealing with an increasing or a decreasing function of x.

(It may be noted that if a series of values of a^x are calculated for values of x forming an arithmetical progression, such as $x=1, 2, 3, \ldots$, the values of a^x form a geometrical progression.)

There are many such quantities in nature.

For example, other things being equal, the population of a country tends to grow at a rate proportional to itself.

Again, a hot body cools at a rate proportional to its excess of temperature above that of surrounding bodies. Also a cistern leaks at a rate proportional to its fulness; and a reservoir of compressed air at a rate proportional to the excess of pressure inside it over the external pressure, And a debt subject to compound interest grows at a rate proportional to itself. This is an artificial growth, by jerks as it were, the interest being added to the principal at finite intervals of time, instead of continuously as in natural growths. But the nature of the growth is similar in the case of all such quantities, whence we may call them all compound interest functions.

Various problems arise in connection with such functions. We will illustrate some of them by examples.

Example. A cup of tea whose temperature 5 minutes ago was 100° above that of surrounding objects, is now 80° above them. Find what its temperature will be in another half-hour, assuming it falls by the compound interest law.

Now, $80^\circ = \frac{4}{5}$ of 100°, so that, in 5 minutes, the temperature drops to $\frac{4}{5}$ of its original value. Similarly, in 5 more minutes the temperature becomes $\frac{4}{5}$ of 80°, *i.e.* $100^\circ \times (\frac{4}{5})^2$, and so on; hence in 35 minutes the temperature $= 100^\circ \times (\cdot 8)^7$, which works out to about 21°, which is the required answer.

The general equation for the excess temperature (T) in t seconds from the time when the temperature was 100° is $T = 100(\cdot 8)^{\frac{t}{5}}$, which is of the form $100a^t$ if $a^5 = \cdot 8$, or $100a^{-t}$, if we write $a^{-5} = \cdot 8$, so as to have a greater than unity, which for some purposes is more convenient.

It will be noticed that $\log T$, which equals $\log 100 - t \log a$, diminishes at a uniform rate, viz. the amount $\log a$ per minute, since $T = 100a^{-t}$, and therefore $\log T = \log 100 - t \log a$. This fact is connected with an important property of compound interest functions, viz. that their *percentage* rate of increase or decrease is constant. This we know is the case with money compound interest. We will prove it generally, and also show how to calculate it.

69. Constant proportional growth of exponential functions. Logarithmic increment or decrement.

Let us suppose that $y = a^x$ where a may be considered as known, or at any rate knowable by means of data supplied by experiment as in the case of the 5 minutes observation of the temperature of the tea in last example.

Then $\frac{dy}{dx} = a^x \log_e a = ky$, say, where $k = \log_e a$.

Now $\frac{dy}{dx}$ is the rate of increase of y per unit increase of x;

and $\frac{1}{y}\cdot\frac{dy}{dx}$ is its rate of increase, per unit of y, per unit increase of x, *i.e.* is its *proportional* rate of growth;

and $\frac{100}{y}\cdot\frac{dy}{dx}$ is its *percentage* rate of growth; *i.e.*, it is the percentage rate of increase of y, per unit increase of x.

To illustrate our meaning more clearly, suppose y to be a sum of money growing continuously by compound interest, and that x is so many years from the commencement of the growth.

Then $\frac{dy}{dx}$ is the actual rate of growth *per annum* of the whole of y,

$\frac{1}{y}\frac{dy}{dx}$ is the rate of interest per £1 per annum,

and $\frac{100}{y}\frac{dy}{dx}$ is the rate of interest per £100 per annum.

[NOTE—*per annum* corresponds to *per unit increase of* x.]

These two latter quantities are constant, but the first of the three grows year by year as y gets larger. It is with the two constant quantities that we have to deal.

The expression $\frac{1}{y}\frac{dy}{dx}$ is the rate of increase per unit of y, and 100 times this is the rate of increase per cent. This last is what we are most used to, but it is more or less artificial, as why should we reckon rate per cent., rather than rate per thousand, or rate per unit? It is in reality merely a numerical convenience, since the rate per unit is generally a small fraction. Of course we do reckon in all three ways as a matter of fact. We talk of 5 %, but the same rate is often spoken of as 1s. in the £, and growth of population is generally reckoned, not per 100, but per 1000. We must consider the rate per *unit* as the natural mode of reckoning rate, and the rates per 100 or per 1000, or per any other number, as merely convenient practical ways of stating the result.

We have now to prove that when $y=a^x$, *i.e.* when y is an exponential or compound interest function, the expression $\frac{1}{y}\frac{dy}{dx}$, which is its *proportional* rate of increase, is constant This follows at once from the equation $\frac{dy}{dx}=ky$, which shows that this *proportional* rate is given by the value of the constant quantity k, which $= \log_e a$.

Hence $\log_e a =$ the rate per *unit* of y, per unit increase of x, and $100 \log_e a$ is the percentage rate of increase of y.

It is interesting to note that $\frac{1}{y}\frac{dy}{dx} = \mathrm{D}\log_e y$.

Hence k, or $\log_e a$, $= \mathrm{D}\log_e y$.

For this reason k, or $\log_e a$, is often called the logarithmic increment of a^x. The percentage rate of increase is then 100 times the logarithmic increment.

If we choose to put a^x in the form of e^{kx}, which we may do, since $a = e^k$, our equation becomes $y = e^{kx}$, and the logarithmic increment is *explicitly* shown by this mode of writing the equation.

Similarly, if $y = a^{-x} = e^{-kx}$, k is called the *logarithmic decrement*, and $100k$ is the percentage rate of *diminution* of y, per unit increase of x.

Example. Determination of the logarithmic decrement in the case of the cup of tea.

The equation connecting T with t is $\mathrm{T} = 100a^{-t} = 100e^{-kt}$, where $k = \log_e a =$ the logarithmic dcerement required.

To evaluate k, we have the equation $a^{-5} = \cdot 8$, so that $a^5 = \frac{10}{8} = 1\cdot 25$, *i.e.* $e^{5k} = 1\cdot 25$.

Hence, taking logarithms, $k = \log_{10} 1\cdot 25 \div 5\mu$, where $\mu = \log_{10} e = \cdot 4343$, and $\frac{1}{\mu} = 2\cdot 3026$.

We shall find k to sufficient accuracy by taking $\frac{1}{5}\log_{10} 1\cdot 25 = 0\cdot 02$, and $\frac{1}{\mu} = 2\cdot 3$, which makes $k = \cdot 046$, and $100k = 4\cdot 6$.

Therefore the fall of temperature is 4·6 per cent. per minute.

The equation for T may be put in the form

$$\mathrm{T} = 100e^{-\cdot 046t},$$

if it is desired to show the value of k explicitly.

70. In the case of money compound interest, we must remember that it is not in practice reckoned continuously, but at intervals, and therefore the above theorems as to rate of interest do not apply without modification.

For example at 5 % compound interest, reckoned annually, the amount in n years is $\mathrm{P}(\mathrm{I} + \frac{1}{20})^n$, if P is the original sum, whereas the continuous compound interest method would give the amount as $\mathrm{P}\,.\,e^{\frac{n}{20}}$, which would be too great. If the accretions of interest were made half-yearly, or quarterly, or at still shorter intervals, the amount would approach more closely to the limit $\mathrm{P}e^{\frac{n}{20}}$, and on the contrary, if the accretions were made biennially or at longer intervals, the amount would approach more nearly to the simple interest limit, viz. :

$$\mathrm{P}\left(1+\frac{n}{20}\right).$$

It will always lie between those two limits.

71. If k is the rate of interest per £ per annum, the amount realized per £ in n years is $(1+k)^n$ if interest is reckoned annually, and is $\left(1+\frac{k}{m}\right)^{mn}$ if interest is reckoned m times a year at equal intervals. If m is infinite this becomes e^{kn}, *i.e.*, the limiting value of $\left(1+\frac{k}{m}\right)^{mn}$ when m is infinite is e^{kn}, and therefore the limiting value of $\left(1+\frac{k}{m}\right)^{\frac{m}{k}}$ is e; or, to put it more simply, $\left(1+\frac{1}{p}\right)^p$ becomes equal to e when $p=\infty$, where p is put for $\frac{m}{k}$ which becomes infinite when m does.

This is sometimes taken as the definition of e.

If we expand $\left(1+\frac{1}{p}\right)^p$ by the binomial theorem, we shall find that when p is infinite, it gives the series already found for e. Thus

$$\left(1+\frac{1}{p}\right)^p=1+p\left(\frac{1}{p}\right)+\frac{p(p-1)}{1\,.\,2}\left(\frac{1}{p}\right)^2+\frac{p(p-1)(p-2)}{1\,.\,2\,.\,3}\left(\frac{1}{p}\right)^3+\dots$$

$$=1+1+\frac{1-\frac{1}{p}}{1\,.\,2}+\frac{\left(1-\frac{1}{p}\right)\left(1-\frac{2}{p}\right)}{1\,.\,2\,.\,3}+\dots,$$

which, when $p=\infty$, becomes $=1+1+\frac{1}{1\,.\,2}+\frac{1}{1\,.\,2\,.\,3}+\dots$, which is the series for e.

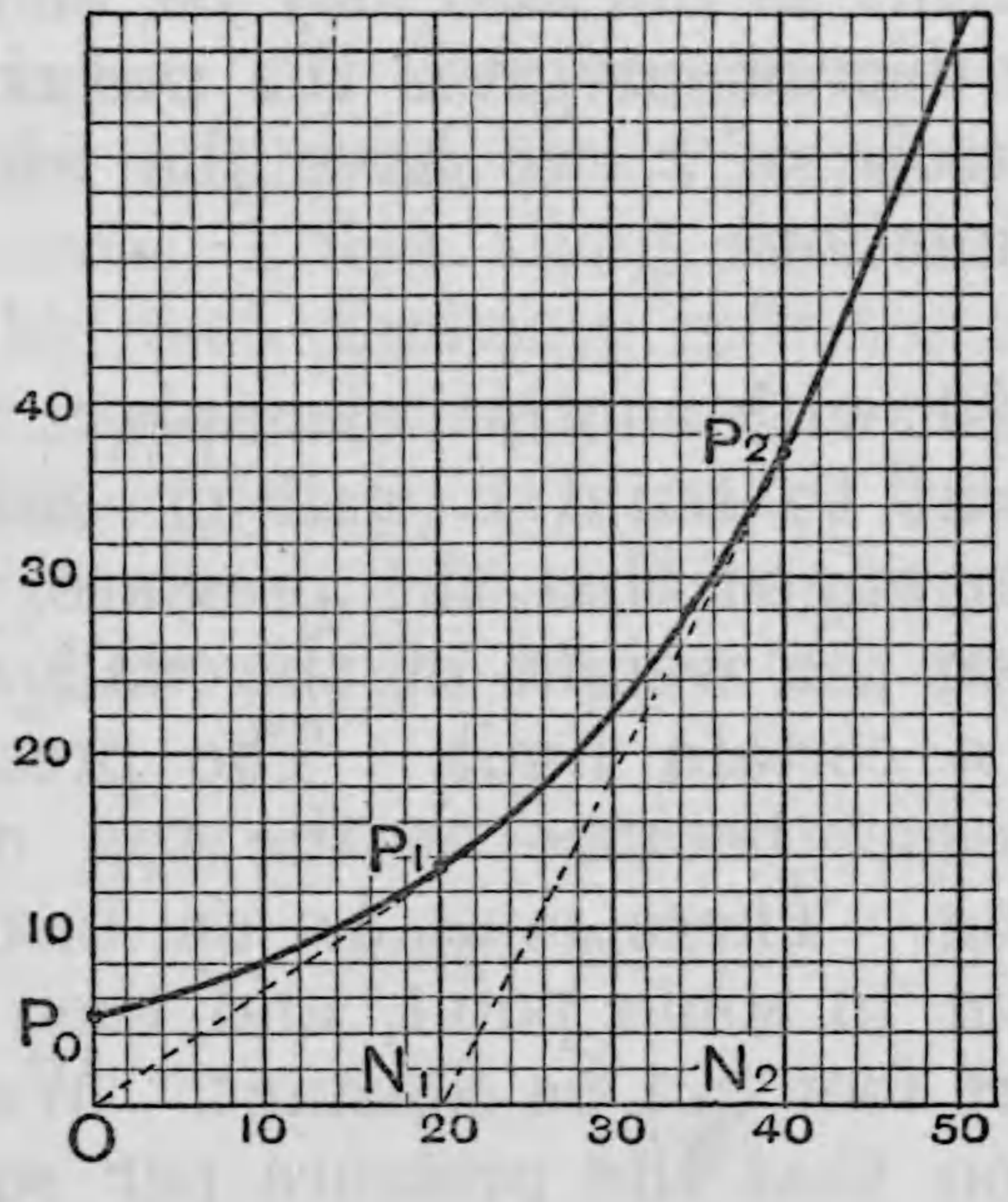
72. *Gradient of the curve* $y = y_0 e^{kx}$.

The gradient is the value of $\frac{dy}{dx}$ at each point, which is equal to $ky_0 e^{kx}$, $= ky$.

To draw the tangent at any point P, whose ordinate is PN, measure from N a length $NN' = \frac{1}{k}$ to the left along the axis of x, then PN′ is the tangent at P. For $PN \div NN' = y \div \frac{1}{k} = ky$, which is the required gradient. The length NN′ intercepted on the axis of x between the ordinate and the tangent is called the subtangent. It is constant and equal to the reciprocal of k in these compound interest curves.*

The easiest way to obtain a rough drawing of the curve is to measure off lengths ON_1, N_1N_2, ... each $= 1/k$ along the axis of x, and to draw the ordinates P_0O, P_1N_1, $P_2N_2 \ldots = y_0, y_1, y_2 \ldots$, where $y_1, y_2 \ldots$ equal ey_0, $e^2y_0 \ldots$ respectively, as is at once seen by putting $x = 1/k$, $2/k$ ... in the given equation. The tangents at P_1, $P_2 \ldots$ will be P_1O, P_2N_1, ...

The following curve is for $y_0 = 5$, and $k = \frac{1}{20}$, *i.e.* $\frac{1}{k} = 20$.



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* A name is much wanted for this reciprocal of k. It might be called the *basic constant*, as it is measured along the base line. When x denotes time, it is called the *time-constant*.

The subjoined table will facilitate the drawing in all such cases:

$$\begin{array}{r|r} e=\ 2{\cdot}72 & e^{-1}={\cdot}368 \\ e^{2}=\ 7{\cdot}39 & e^{-2}={\cdot}135 \\ e^{3}=20{\cdot}08 & e^{-3}={\cdot}050 \\ e^{4}=54{\cdot}60 & e^{-4}={\cdot}018 \end{array}$$

The student might draw the decreasing curve given by the equation

$$y=100e^{-\frac{x}{20}}$$

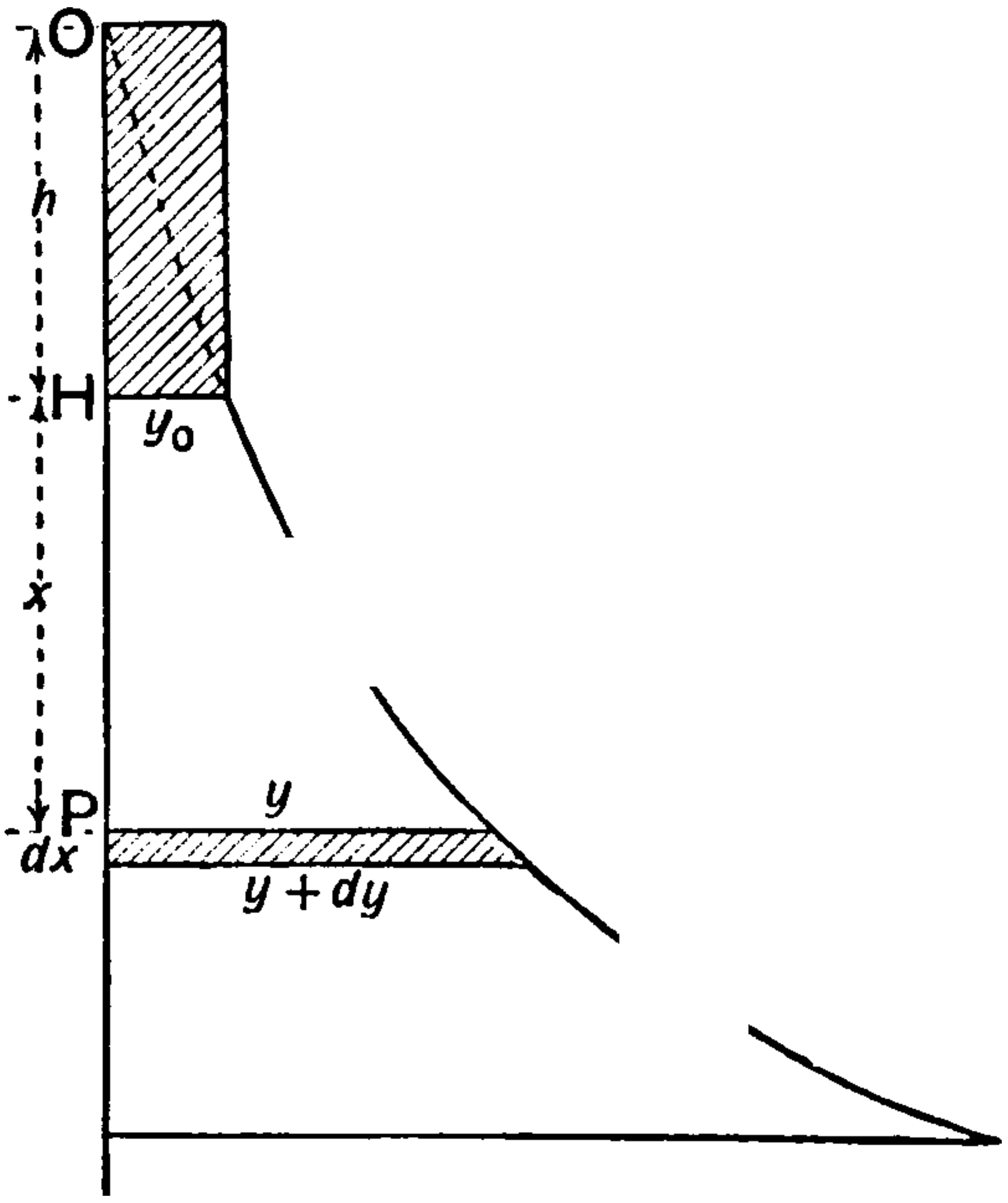
in the same way. The successive ordinates will be each $\frac{1}{e}$ of the preceding one, and the tangents will be P_0N_1, P_1N_2, ...

73. If many ordinates are wanted, and particularly if the equation is given in the form $y=y_0a^{\pm x}$, it is probably best to use a table of logarithms, and to work with the equation $\log_{10}y=\log_{10}y_0\pm Ax$, where $A=\log_{10}a$. If it is also desired to draw any of the tangents, the value of the subtangent $\frac{1}{k}$ can be readily found, being equal to $\frac{\mu}{A}$. If it is remembered that this is the reciprocal of the logarithmic increment or logarithmic decrement as the case may be, and that 100 times this increment or decrement gives the percentage rise or fall of y per unit increase of x, we have the whole theory and practice of these functions.

74. We will finish with another example.

Suppose we want to build a wall or embankment with cross-sections so arranged that the pressure per square foot at any part due to the weight of the wall above that part shall not exceed a certain limit. The pressure cannot be everywhere the same, because at the top of the wall the pressure is nothing. There must be an increasing pressure from the top down to some point, and then below this the pressure per square foot can be constant. We must therefore make the condition that the pressure per square foot at a certain point some distance below the top shall not be anywhere exceeded. We will suppose the wall to be vertical on one side and battered on the other.

Let the pressure at H in the figure $=p$ tons per square foot, and let it be required that the pressure per square foot at any distance x below H shall be the same as at H. Let the material used weigh w tons per cubic foot. Let y feet be the thickness of the wall at this depth (at the point P). Then the weight resting on a horizontal section at P, y feet across × 1 along, will, on our supposition, be py tons. Now take a horizontal section 1 foot along and $y+dy$ feet across



at a depth dx below P. The weight resting on this will be greater than the weight on the P section by the weight of a slab y feet across, 1 foot along, and dx feet deep, *i.e.* containing ydx cubic feet (omitting infinitesimals of the second order). The increase of weight is therefore $wydx$ tons, and the total weight $=(py+wydx)$ tons. But the area of the section is now $y+dy$ sq. feet, and as the pressure per sq. foot is still to be $=p$, the weight supported must $=p(y+dy)$ tons. Hence

$$py+wydx=p(y+dy);$$

$$\therefore\ wydx=pdy;$$

$$\therefore\ \frac{dy}{dx}=\frac{w}{p}y.$$

∴ the thickness y of the wall follows the compound interest law, since its logarithmic increment is constant.

Let $\frac{w}{p}=k$, then $\frac{dy}{dx}=ky$, and $y=y_0e^{kx}$, where y_0 is the thickness at H.

We can put this result in a simpler form if we suppose the wall above H to be of uniform thickness. Let h feet be the height OH of this part. Then the weight per square foot resting on the horizontal section at H is evidently hw. Hence

$$hw=p;$$

$$\therefore\ y=y_0e^{\frac{x}{h}}.$$

If, then we take points H_1, H_2, ... below H, such that $HH_1=H_1H_2= \dots =h$, the thickness at these points will be ey_0, e^2y_0, ... and the tangents to the battered slope of the wall at these levels will pass through the points H, H_1 ..., and, similarly, the tangent at the level of H will pass through O.

If we wish to obtain intermediate thicknesses by calculation, the easiest plan is to calculate $e^{\frac{1}{h}}$, or rather its $\log_{10}$, and then to calculate the values of y from the equation

$$\log_{10}y=\log_{10}y_0+\text{A}x,\ \text{when}\ \text{A}=\log_{10}e^{\frac{1}{h}}=\frac{\mu}{h}.$$

Suppose, for example $y_0=1$, and $h=10$ (measured in feet),

then $$\frac{\mu}{h}=\frac{\cdot 4343}{10}=\cdot 04343$$

$$\text{and}\ \log_{10}y=\cdot 04343x.$$

If this wall is 30 feet high, *i.e.* extends 20 feet below H, the thickness at the base will be 7·4 feet. If it extends 30 feet below H, its base must be 20 feet across.

If the wall is battered on both sides, the widths at the various levels will be just the same as in the case considered. For the width depends on the superincumbent weight, and not on the relative amounts of batter.

NOTE.—If the upper part OH of the wall is not of uniform thickness, the easiest plan will be to take as h, not its height, but the height of an equivalent weight of wall that is of uniform thickness equal to the thickness at H. The equations will then hold without modification.

75. The function $e^{-ax}\sin(bx+c)$.

This function is an oscillating function with decreasing amplitude. It is connected with such problems as the extent of swing of a pendulum which is set swinging in a resisting medium and left to come gradually to rest. It is important also in connection with many electrical problems.

The student should draw the graph of this function for different values of a, b and c. (See examples Nos. 9-13, p. 75).

The function crosses the zero line ($y=0$) when $bx+c=0$, π, 2π, ... Its graph oscillates between the curves $y=e^{-ax}$ and $y=-e^{-ax}$, touching them alternately when $bx+c=\frac{1}{2}\pi$, $\frac{3}{2}\pi$, $\frac{5}{2}\pi$, ...

$$\text{Its gradient} = e^{-ax}[b\cos(bx+c) - a\sin(bx+c)]$$
$$= -\sqrt{a^2+b^2}\,e^{-ax}\sin(bx+c-\alpha),$$

where $\tan\alpha = \dfrac{b}{a}$.

Its gradient is therefore zero when $bx+c=\alpha$, $\alpha+\pi$, $\alpha+2\pi$,....

76. Recapitulation.

$$d(e^x) = e^x dx,$$
$$d(a^x) = a^x \log_e a\, dx,$$

where $$e^x = 1 + x + \frac{x^2}{1\,.\,2} + \frac{x^3}{1\,.\,2\,.\,3} + \dots,$$

whence $$e = 1 + 1 + \frac{1}{1\,.\,2} + \frac{1}{1\,.\,2\,.\,3} + \dots$$
$$= 2{\cdot}718281828\dots.$$
$$\log_{10} e = \mu = 0{\cdot}43429448\dots,$$

and $$\log_e 10 = \frac{1}{\mu} = 2{\cdot}30258509\dots.$$

e is also the limit of $\left(1+\dfrac{1}{p}\right)^p$ when $p=\infty$,

which is the same as the limit of $(1+h)^{\frac{1}{h}}$ when $h=0$.

The limit of $\dfrac{e^h-1}{h}$ is 1, when $h=0$;

and the limit of $\dfrac{a^h-1}{h}$ is $\log_e a$, when $h=0$.

$d\log_e x = \frac{dx}{x}$, and, of course, $d\log_e u = \frac{du}{u}$, often used in the form

$$du = u\,d\log_e u, \text{ where } u \text{ is any function of } x.$$

$d\log_a x = \log_a e \cdot \frac{dx}{x}$; and, in particular, $d\log_{10} x = \mu\frac{dx}{x}$.

If $\quad y = y_0 a^{\pm x} = y_0 e^{\pm kx}$; $\frac{dy}{dx} = \pm ky$, and $\frac{1}{y}\frac{dy}{dx} = \pm k$, *i.e.* y grows or diminishes at a rate proportional to itself, *i.e.* by the law of compound interest; and $100k$ is the constant rate *per cent.* at which it grows or diminishes, k being called the logarithmic increment or decrement of the function.

The curve has the constant subtangent $\frac{1}{k}$.

Corresponding to the values $x = 0, \frac{1}{k}, \frac{2}{k}, \ldots$ the values of y are $y_0, ey_0, e^2y_0, \ldots$ or $y_0, \frac{y_0}{e}, \frac{y_0}{e^2}, \ldots$ according to whether y is increasing or decreasing, and these values, with the gradients at the points, will often suffice for a rough drawing.

If we require intermediate values of y, the working equation is $\log_{10} y = \log_{10} y_0 \pm \mathrm{A}x$, where $\mathrm{A} = \mu k = \log_{10} a$.

EXAMPLES.

1. Expand $\left(1+\frac{1}{p}\right)^{px}$ by the Binomial theorem, and by putting $p = \infty$ in the result, deduce the series for e^x. Show that $\left(1+\frac{x}{p}\right)^p$ has the same limiting value.

2. Differentiate $\left(1+\frac{x}{p}\right)^p$, and deduce the fact that $de^x = e^x dx$.

3. If the population of a county doubles itself in 100 years, find its rate of growth per 1000 per annum, assuming it constant. If the population is a million at the beginning of the century, find what it will be in 20, 50, and 80 years respectively from the beginning.

4. Supposing that if a cable is coiled once round a post a pull on it equal to 50 lbs. weight will balance a resistance of 500 lbs. weight, find what resistance can be balanced by the same pull if the rope is coiled three times round, supposing that the rope is strong enough, and that the advantage of the greater number of turns increases by the compound interest law.

5. In the last question find the equation connecting the pull and resistance in the form $R=P.a^{\theta}$, where θ is measured (1) in whole revolutions, (2) in radians, (3) in degrees.

6. Find the ratio of R to P (1) if a half-turn is taken, (2) if only a quarter-turn is taken.

7. Find the equation in the form $R=Pe^{k\theta}$ when θ is in radians.

8. Calculate the thickness at the base of a wall 50 feet high if the top 20 feet have a uniform batter, the thickness of this part increasing from 1 foot at the top to 2 feet at the bottom, and the lower part of the wall being so built that the intensity of pressure at every part of it is the same.

Examples of $e^{-ax}\sin bx$:

9. Draw the curve $y=e^{-\frac{1}{2}x}\sin \pi x$.

10. Draw the curve $y=e^{-\frac{1}{2}x}\sin 2\pi x$.

11. Draw the curve $y=e^{-\frac{1}{2}x}\sin\left(\pi x+\frac{\pi}{2}\right)$.

12. Draw the curve $y=e^{-\frac{1}{2}x}\sin\left(\frac{1}{10}\pi x+\frac{\pi}{2}\right)$.

13. Draw the curve $y=e^{-\frac{1}{2}x}\sin(\frac{1}{10}\pi x)$.

[In the curves 9-13 find specially where the curves touch the guiding curves $y=\pm e^{-\frac{1}{2}x}$, which should be drawn first; also where they cut the zero line, and at what angles; also where their gradients are zero.]

Further compound interest problems:

14. A pane of glass destroys or obliterates 5 per cent. of the light falling upon it; how much light gets through 20 such panes one behind the other, assuming that they all act in the same way?

15. In a certain rarefied gas one per cent. of the molecules which start in any given direction are deflected by collision with others after travelling a millimetre; how many are able to travel a centimetre without collision?

16. An electric current left to die out in a certain circuit drops to $\frac{1}{e}$ of its value in $\frac{1}{10}$ of a second; how long will it take to drop to a millionth of its value, assuming that it decreases at a rate proportional to itself?

17. A current left to itself in a very massive conductor diminishes only ten per cent. in five seconds: how soon will it become imperceptible on a galvanometer which can detect a billionth part (10^{-12}) of the initial current?

18. A current is started in the same conductor: find how soon it will rise to within one-thousandth part of its full value, assuming that it rises at a rate proportional to its defect from the maximum? Find the time-constant of this conductor, and draw the curves of rise and fall of the current.

CHAPTER V.

HYPERBOLIC FUNCTIONS.

77. There have come into use of late years certain functions of e whose properties are very similar in many respects to those of the circular functions. They are connected with a rectangular hyperbola very much in the same way as the circular functions are connected with the circle—hence their name. We shall however here simply give their definitions and more important properties without reference to their hyperbolic connection.

The most important of these are

(1) $\frac{1}{2}(e^x + e^{-x})$ which is called the hyperbolic cosine of x, or briefly, $\cosh x$, or $\operatorname{ch} x$, and

(2) $\frac{1}{2}(e^x - e^{-x})$, called the hyperbolic sine of x, or $\sinh x$, or $\operatorname{sh} x$.

Besides these there are* $\tanh x\left(=\dfrac{\sinh x}{\cosh x}\right)$, and its reciprocal $\coth x$; and $\operatorname{sech} x$, which is the reciprocal of $\cosh x$; and lastly, $\operatorname{cosech} x$, which is the reciprocal of $\sinh x$.

78. Since $$e^x = 1 + x + \frac{x^2}{1\,.\,2} + \frac{x^3}{1\,.\,2\,.\,3} + \dots$$

and, therefore, $$e^{-x} = 1 - x + \frac{x^2}{1\,.\,2} - \frac{x^3}{1\,.\,2\,.\,3} + \dots$$

---

* The student should draw careful graphs of $\sinh x$, $\cosh x$ and $\tanh x$ in one diagram, so as to get an idea of their general aspect. The shape of $\cosh x$ is that of a chain hanging between two supports.

it follows that $\cosh x = 1 + \frac{x^2}{1 \,.\, 2} + \frac{x^4}{1 \,.\, 2 \,.\, 3 \,.\, 4} + \ldots$

and $\sinh x = x + \frac{x^3}{1 \,.\, 2 \,.\, 3} + \frac{x^5}{1 \,.\, 2 \,.\, 3 \,.\, 4 \,.\, 5} + \ldots;$

expansions very similar to those of $\cos x$ and $\sin x$, except that all the terms are positive instead of being alternately + and −.

79. The student can, from the definitions of Art. 77, very readily prove the following properties, and should compare and contrast them with the corresponding properties of the circular functions.

(1) $d \cosh x = \sinh x \, dx.$

(2) $d \sinh x = \cosh x \, dx.$

(3) $\cosh^2 x - \sinh^2 x = 1.$

(4) $\cosh^2 x + \sinh^2 x = \cosh 2x.$

(5) $2 \sinh x \cosh x = \sinh 2x.$

(6) $d \tanh x = \operatorname{sech}^2 x \, dx.$

(7) $d \coth x = -\operatorname{cosech}^2 x \, dx.$

(8) $d \operatorname{sech} x = -\operatorname{sech} x \tanh x \, dx.$

(9) $d \operatorname{cosech} x = -\operatorname{cosech} x \coth x \, dx.$

(10) $\tanh^2 x + \operatorname{sech}^2 x = 1.$

(11) $\coth^2 x - \operatorname{cosech}^2 x = 1.$

And last, but by no means least,

(12) $x = \log_e(\cosh x + \sinh x).$

The 3rd and 12th formulae give rise to the inverse formulae:

(13) $\sinh^{-1} x = \log_e[x + \sqrt{(x^2 + 1)}]$ $\left\{\begin{array}{l}\text{proved by putting} \\ \qquad x = \sinh u, \\ \text{whence} \\ \qquad \cosh u = \sqrt{(x^2+1)}, \\ \qquad \therefore\ u = \text{etc.}\end{array}\right.$

$$(14)\quad \cosh^{-1}x=\log_e[x+\sqrt{(x^2-1)}]\ \left\{\begin{array}{l}\text{proved by putting}\\ \qquad\qquad x=\cosh u,\\ \text{whence}\\ \qquad\quad \sinh u=\sqrt{(x^2-1)},\\ \qquad \therefore\ u=\text{etc.}\end{array}\right.$$

$$(15)\quad \tanh^{-1}x=\tfrac{1}{2}\log_e\frac{1+x}{1-x}\qquad \left\{\begin{array}{l}\text{proved by putting}\\ \qquad\qquad x=\tanh u,\\ \text{and simplifying the right-}\\ \text{hand side.}\end{array}\right.$$

The most important of these formulae are those relating to $\cosh x$ and $\sinh x$, viz. (1) to (5), and (12), together with the two inverse formulae (13) and (14).

The student should draw the graphs of $\cosh x$ and $\sinh x$, which he can readily do by means of the little table on p. 70. The curve $y=\cosh x$ is the shape of a uniform chain hanging between two supports, and is therefore of special interest. It is called the common catenary, or, briefly, the catenary.

80. Formula (3), viz. $\cosh^2x-\sinh^2x=1$, is of great importance in many ways. It is similar in form to the identity

$$\sec^2\theta-\tan^2\theta=1,$$

and therefore we are able to make use of a comparative table of secants and tangents if we wish to find $\sinh x$ when we know $\cosh x$, or *vice versa*.

Moreover, if we equate $\cosh x$ to $\sec\theta$, and so make

$$\sinh x=\tan\theta,$$

it follows that

$$\tanh x=\frac{\tan\theta}{\sec\theta}=\sin\theta;$$

also

$$\coth x=\operatorname{cosec}\theta,$$

$$\operatorname{sech} x=\cos\theta,$$

and

$$\operatorname{cosech} x=\cot\theta.$$

Hence the relations are:

$\sin\theta$	$\operatorname{cosec}\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\cos\theta$
$\tanh x$	$\coth x$	$\sinh x$	$\operatorname{cosech} x$	$\cosh x$	$\operatorname{sech} x$

which are perhaps best put in the following three groups:

$\sin\theta$	$\tan\theta$	$\sec\theta$	$\cos\theta$	$\operatorname{cosec}\theta$	$\cot\theta$
$\tanh x$	$\sinh x$	$\cosh x$	$\operatorname{sech} x$	$\coth x$	$\operatorname{cosech} x$

Of these the quantities in the third group are merely the reciprocals of those in the first, while those in the second group are mutually reciprocal.

81. The function $\lambda(\theta)$.

From formula (12) we see that if $\cosh x = \sec\theta$, and $\sinh x = \tan\theta$, the value of x itself is given by the equation

$$x = \log_e(\sec\theta + \tan\theta).$$

This is a most important function of θ. It comes into very many formulae in mechanics and other applications of mathematics. In particular the spacing of the parallels of latitude in Mercator's projection is connected with this function, their distances from the equator being given by the equation

$$y = a\log_e(\sec\theta + \tan\theta)$$

for different values of the latitude θ, a being the constant which is used in spacing the lines of longitude from the zero meridian by the equation $x = a\phi$ for different values of the longitude ϕ.

The importance of this function has led certain French writers to give it a special name. They call it *lambda* (θ), writing it $\lambda(\theta)$.

We may therefore state the connection between x and θ by saying that if $x = \lambda(\theta)$, then $\cosh x = \sec\theta$, $\sinh x = \tan\theta$, etc. θ is sometimes called the gudermannian of x, from Gudermann, who first systematically studied these functions, but the name is not very important, as we generally think of x as a function of θ rather than θ as a function of x.

82. *Calculation of* $\lambda(\theta)$.

It is easy to prove that $\sec\theta+\tan\theta=\tan(45^\circ+\frac{1}{2}\theta)$.

For
$$\sec\theta+\tan\theta=\frac{1+\sin\theta}{\cos\theta}=\frac{\sin 90^\circ+\sin\theta}{\cos 90^\circ+\cos\theta}$$
$$=\frac{2\sin(45^\circ+\frac{1}{2}\theta)\cos(45^\circ-\frac{1}{2}\theta)}{2\cos(45^\circ+\frac{1}{2}\theta)\cos(45^\circ-\frac{1}{2}\theta)}$$
$$=\tan(45^\circ+\tfrac{1}{2}\theta).$$

Hence
$$\lambda(\theta)=\log_e\tan(45^\circ+\tfrac{1}{2}\theta)$$
$$=\frac{1}{\mu}\log_{10}\tan(45^\circ+\tfrac{1}{2}\theta),$$

where $\frac{1}{\mu}=2{\cdot}302585$.

Therefore, to calculate $\lambda(\theta)$, look out $\log_{10}\tan(45^\circ+\frac{1}{2}\theta)$ and multiply it by $\frac{1}{\mu}$, which can readily be done by logarithms, or by help of a multiplication table provided for the purpose in most books of tables. [For a short table of values of $\lambda(\theta)$ see p. 82.]

We have, then, to remember that
$$\lambda(\theta)=\log_e(\sec\theta+\tan\theta)=\log_e\tan(45^\circ+\tfrac{1}{2}\theta).$$

83. *The differential of* $\lambda(\theta)$.

$$d\,\lambda(\theta)=\frac{d(\sec\theta+\tan\theta)}{\sec\theta+\tan\theta}=\frac{(\sec\theta\tan\theta+\sec^2\theta)d\theta}{\sec\theta+\tan\theta}$$
$$=\sec\theta\,.\,d\theta.$$

84. The curve $y=\lambda(\theta)$ will be found instructive. It has real and imaginary regions, being real when $\cos\theta$ is positive and imaginary when $\cos\theta$ is negative.

For
$$\sec\theta+\tan\theta=\frac{1+\sin\theta}{\cos\theta}.$$

The numerator of this is always positive; therefore the fraction is + or − according as $\cos\theta$ is + or −. There is no real logarithm of a negative quantity; hence $\lambda(\theta)$ is real when $\cos\theta$ is positive, and only then, that is, when θ is between $-90°$ and $+90°$ or between $n360°-90°$ and $n360°+90°$. Hence we need only draw the graph between $\theta=-90°$ and $+90°$, as the graph for other values of θ will be merely a repetition of this part.

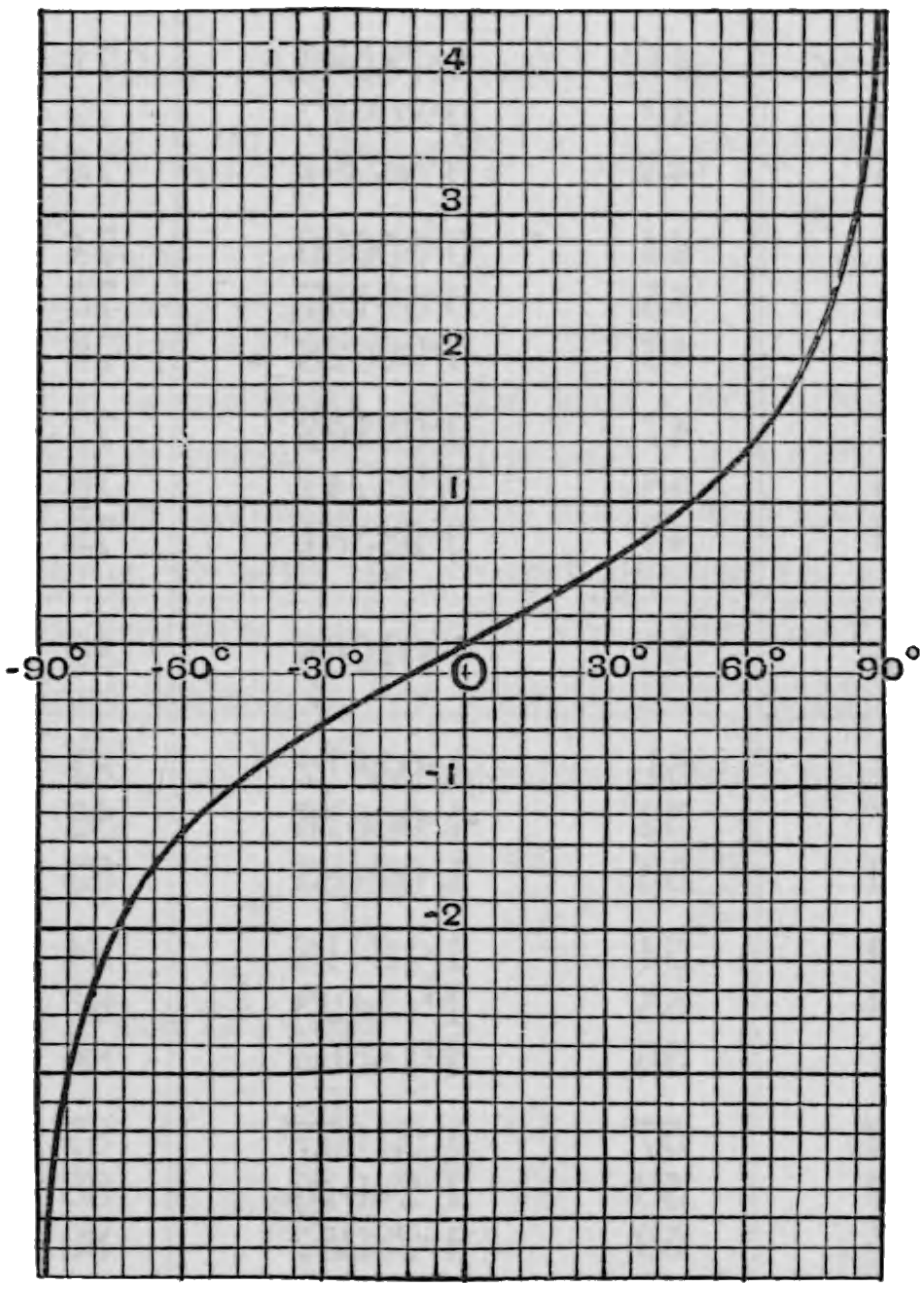
To draw the graph for negative values of θ, we should notice that

$$\lambda(-\theta)=-\lambda(\theta).$$

For $$\lambda(-\theta)=\log_e(\sec\theta-\tan\theta)$$

and $$\lambda(\theta)=\log_e(\sec\theta+\tan\theta);$$

$$\therefore\ \lambda(-\theta)+\lambda(\theta)=\log_e(\sec^2\theta-\tan^2\theta)$$
$$=\log_e 1=0.$$



If we then first draw the graph from 0° to 90°, by help of the table, it is easy to draw the part from 0 to $-90°$, as it is merely the former part reversed in direction. (See figure.)

$$u=\log_e(\sec\theta+\tan\theta)=\log_e\tan(45^\circ+\tfrac{1}{2}\theta).$$

θ	u	θ.	u	θ	u
0°	0·00000	30°	0·54931	60°	1·31696
1	0·01745	31	0·56956	61	1·35240
2	0·03491	32	0·59003	62	1·38899
3	0·05238	33	0·61073	63	1·42679
4	0·06987	34	0·63166	64	1·46591
5	0·08738	35	0·65284	65	1·50685
6	0·10491	36	0·67428	66	1·54855
7	0·12248	37	0·69599	67	1·59232
8	0·14008	38	0·71699	68	1·63794
9	0·15773	39	0·74029	69	1·68557
10	0·17543	40	0·76291	70	1·73542
11	0·19318	41	0·78586	71	1·78771
12	0·21099	42	0·80917	72	1·84273
13	0·22886	43	0·83284	73	1·90079
14	0·24681	44	0·85690	74	1·96226
15	0·26484	45	0·88137	75	2·02759
16	0·28295	46	0·90628	76	2·09732
17	0·30116	47	0·93163	77	2·17212
18	0·31946	48	0·95747	78	2·25280
19	0·33786	49	0·98381	79	2·34040
20	0·35638	50	1·01068	80	2·43625
21	0·37501	51	1·03812	81	2·54209
22	0·39377	52	1·06616	82	2·66031
23	0·41266	53	1·09483	83	2·79422
24	0·43169	54	1·12418	84	2·94870
25	0·45088	55	1·15423	85	3·13130
26	0·47021	56	1·18505	86	3·35467
27	0·48872	57	1·21667	87	3·64253
28	0·50939	58	1·24916	88	4·04813
29	0·52925	59	1·28257	89	4·74135
30	0·54931	60	1·31696	90	infinite

To find u for intermediate values of θ, it will be best to look out $\log_{10}\tan(45^\circ+\frac{1}{2}\theta)$ and multiply it by 2·302585.

EXAMPLES.

Differentiate the following functions:

1. $\cosh x \cos x + \sinh x \sin x$.
2. $\cosh x \sin x + \sinh x \cos x$.
3. $\log \cosh x$.
4. $\sin^{-1}(\tanh x) + \tan^{-1}(\sinh x)$.

5. If ϕ is the gradient angle at any point of the curve $y = \cosh x$, show that $y = \sec\phi$ and $x = \lambda(\phi)$.

6. Prove that $2\cosh^2 x = \cosh 2x + 1$
and $2\sinh^2 x = \cosh 2x - 1$.

7. Prove that $4\cosh^3 x = \cosh 3x + 3\cosh x$.

8. Prove that $4\sinh^3 x = \sinh 3x - 3\sinh x$.

9. Show that differentiating the identity of Ex. 7 will give us the identity of Ex. 8, and *vice versa*.

MISCELLANEOUS EXAMPLES.

Find an expression for $\dfrac{dy}{dx}$ in the following five examples:

1. $x^2 - xy + 2y^2 - 2ax - 6ay + 7a^2 = 0$.
2. $y = \tan^{-1}x + \frac{1}{2}\log\dfrac{1+x}{1-x}$.
3. $y = \sin(e^x \log x) \cdot \sqrt{\{1 - (\log x)^2\}}$.
4. $y = e^{-ax}\cos 3x$.
5. $y = x^2 \sin 2x$.

6. If $u = \dfrac{q^2 r^2}{(p^2+q^2)(p^2+r^2)}$,
find du (1) if p is variable, and q and r are constants;
(2) if q is variable, and p and r are constants;
(3) if r is variable, and p and q are constants.

[Probably at first it will be found easier to substitute x for the variable, and then replace the proper variable after differentiating.]

7. If in the last question we denote the respective differential coefficients by $\dfrac{du}{dp}$, $\dfrac{du}{dq}$, and $\dfrac{du}{dr}$ respectively, find the value of

$$p \cdot \frac{du}{dp} + q \cdot \frac{du}{dq} + r \cdot \frac{du}{dr}.$$

8. If $r^2\cos^2\theta = a^2\cos 2\theta$, find $\dfrac{dr}{d\theta}$.

9. If $u = \cos^{-1}x - \dfrac{2(1-x)^{\frac{1}{2}}}{(1+x)^{\frac{1}{2}}}$, find $\dfrac{du}{dx}$.

10. If $x = a(\theta - \sin\theta)$, and $y = a(1 - \cos\theta)$, find $\dfrac{dy}{dx}$.

11. If $y = \dfrac{3x^2 - 4z^2}{(4x - 7z)^4}$, find $\dfrac{dy}{dx}$ when z is constant, and find $\dfrac{dy}{dz}$ when x is constant, and deduce the value of $x\dfrac{dy}{dx} + z\dfrac{dy}{dz}$.

CHAPTER VI.

APPROXIMATE DETERMINATION OF THE ROOTS OF EQUATIONS.

85. General methods of approximating to the roots of equations containing one unknown quantity.

In solving a mathematical problem it is often necessary to find the value or values of an unknown quantity which will satisfy some equation. This is called "finding the root or roots of the equation," or "solving the equation." The student has doubtless often done this work in connection with simple and quadratic equations in one unknown quantity. The object of this chapter is to give methods by which the roots of any equation in one unknown quantity can be readily found, either accurately or approximately. Since all the terms of an equation can, if desired, be put on the left-hand side of the sign of equality, with *zero* on the right-hand side, the general type of such equations is $f(x)=0$, where $f(x)$ denotes some function of x. In the case of algebraic equations, it is sometimes possible to split $f(x)$ into factors; in such case the roots are easily found. Thus, if $x^2-3x+2=0$, we see that $(x-2)(x-1)=0$, $\therefore$ the roots are $x=2$ and $x=1$. This may be called "solution by factorization." This method is only possible in simple cases, and can very seldom be used in solving equations of higher degree than quadratics, and only occasionally even in quadratics. The general method of solving quadratics is, however, well known.

The methods which will be suggested in this chapter are applicable to equations of any degree, and to transcendental as well as to algebraic equations.

These methods may be divided into two main classes, namely, *graphic* and *by calculation*. These two methods can be made to mutually help each other, and the graphic method will certainly help the student to understand the method by calculation. My object will be, not to attempt in any way an exhaustive treatment of the subject, but merely to give ordinary methods which will be applicable to any equation. I shall assume in the graphic work that the student is able to obtain squared paper to facilitate plotting.

86. Graphic method—First approximation.

Every equation in x can be put into the form $f(x)=0$. If then we plot the curve $y=f(x)$, and find where this curve cuts the axis of x, we shall have found the roots of the equation. In general we shall not be able to find the roots exactly, but we shall be able to find two near values of x between which a root lies. We shall show how to get nearer and nearer approximations to the root, till any desired accuracy has been obtained.

Example. Find the roots of the equation $x^3-4x^2-11x+32=0$. To plot the curve $y=f(x)$ we shall find the values of y for different values of x, and to ensure correct drawing at points near to the axis of x, we will also calculate the gradient at such points. The gradient is given by the equation $\tan\phi=f'(x)=3x^2-8x-11$.

Positive values of x.

x	y	$\tan\phi$	Points.
0	+32		
1	+18		
2	+ 2	−15	A
3	−10	− 8	B
4	−12	+ 5	C
5	+ 2	+24	D

etc.

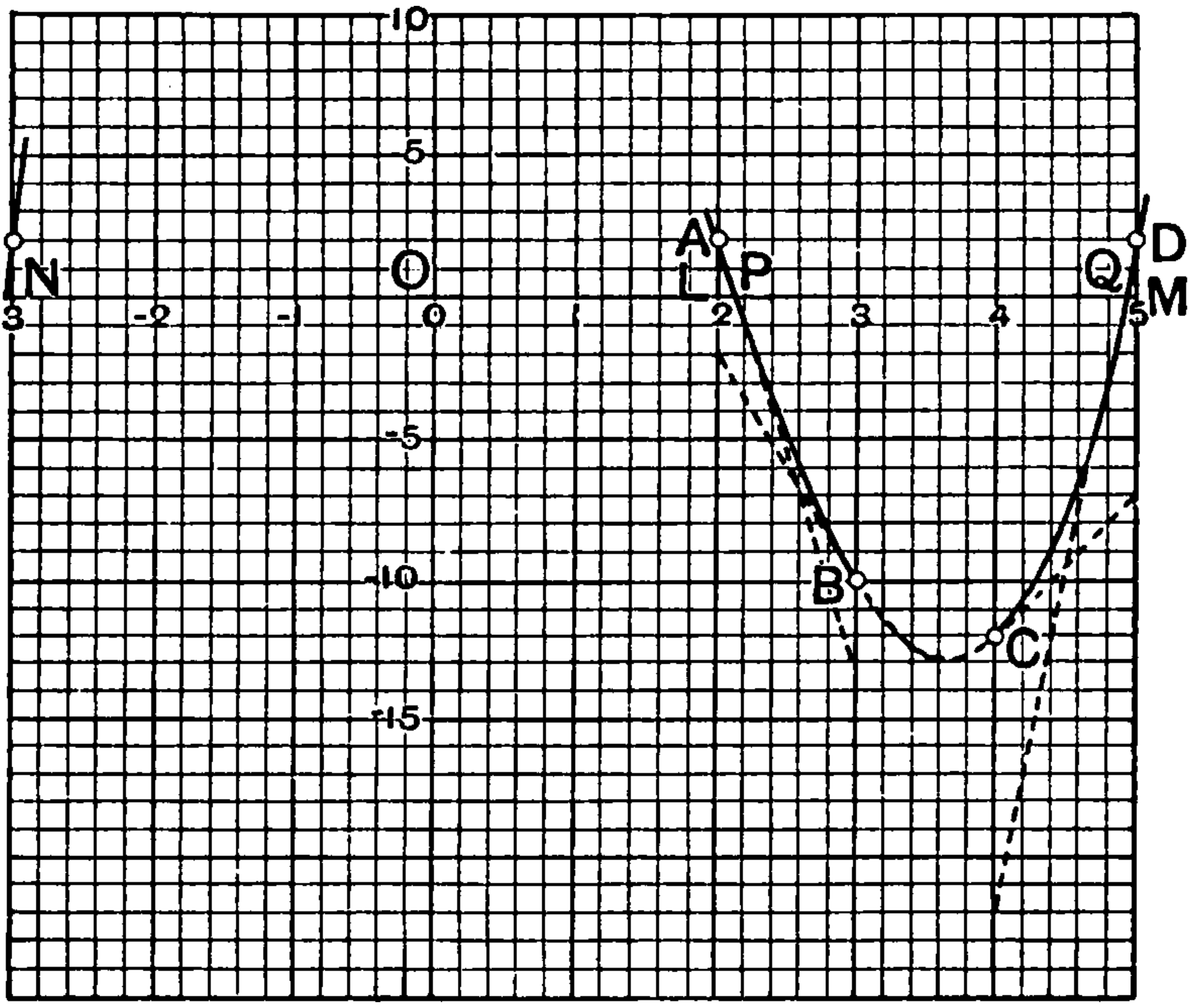
Negative values of x.

x	y	$\tan\phi$	Points.
0	+32		
−1	+38		
−2	+30		
−3	+ 2	+40	E
−4	−52	+69	F

[not shown in diagram, being too far down.

This shows that there is a root between 2 and 3, but nearer 2 than 3; because $y=+2$ when $x=2$, and $y=-10$ when $x=3$, whence y must $=0$ somewhere between.

Similarly there is a root between 4 and 5, but nearer to 5 than 4; and another root between -3 and -4, but much nearer to -3 than to -4. If we draw the parts of the curve to scale in the neighbourhood of the above points, we shall be able to judge roughly the values of the roots. In the diagram the x-scale is $\frac{1}{2}$ inch to a unit, and the y-scale is $\frac{1}{10}$ inch to a unit. The curve is drawn through A, B, C, D, and E by help of the gradient values calculated in the table, and the roots are approximately between 2·1 and 2·2, between 4·8 and 5·0, and between $-3{\cdot}0$ and $-3{\cdot}1$.



87. Further approximation, by calculation and graphically. It will be noticed that in each of the above cases the curve and the tangent are exceedingly close together as they cross the zero line. If we take the point where the tangent crosses as being sufficiently near the true point, we can readily do the calculation. Take, for example, the root between A, B. Let the tangent at A cut the axis in P; the ordinate AL is $=2$, and the gradient of AP is $\tan\phi_1 = -15$. Hence $\text{LP} = -\text{AL}\cot\phi = \frac{2}{15}$;

$$\therefore \text{ the length of } \text{OP} = \text{OL} - \text{AL}\cot\phi_1 = 2 + \tfrac{2}{15}, \textit{ i.e. } 2{\cdot}13 \ldots.$$

Similarly the other positive root

$$= \text{OQ} = \text{OM} - \text{DM}\cot\phi_2 = 5 - \tfrac{2}{24} = 4{\cdot}92,$$

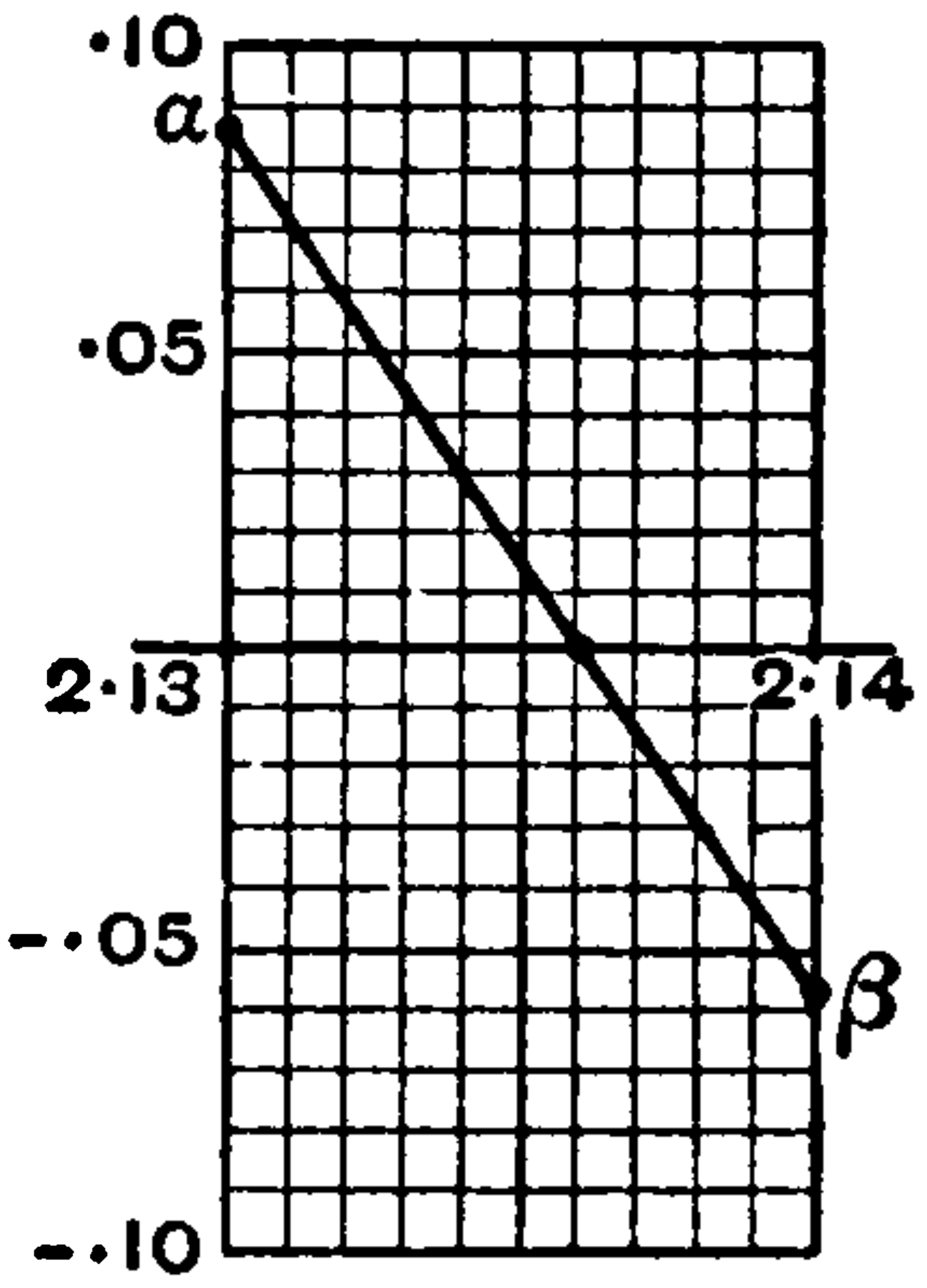
and the negative root $= -3 - \frac{2}{40} = -3{\cdot}05$.

If these values are not accurate enough for the purpose we need, we can further approximate by using a larger scale figure (working at each root separately), or by calculation, or both ways; this indeed being the best.

(1) Take the root near OP.

When $x = 2{\cdot}13,\quad f(x) = +0{\cdot}085997,\quad f'(x) = -14{\cdot}43;$

and when $x = 2{\cdot}14,\quad f(x) = -0{\cdot}058056,\quad f'(x) = -14{\cdot}38;$



so that we have approximately the points

$$\alpha,\quad (2{\cdot}13,\quad 0{\cdot}086)$$

and

$$\beta,\quad (2{\cdot}14,\quad -0{\cdot}058).$$

The curve between these points will be so nearly straight, that we may take the crossing point of the chord $\alpha\beta$ as near enough to the true point. This gives, graphically, 2·1360 for the root.

To check this by calculation by means of the tangent at α; the gradient at α is $-14{\cdot}43$:

$$\therefore\ x = 2{\cdot}13 + \frac{0{\cdot}085997}{14{\cdot}43} = 2{\cdot}13596;$$

or we might have taken the gradient at β, which is $-14{\cdot}38$,

whence $$x = 2{\cdot}14 - \frac{0{\cdot}058056}{14{\cdot}38} = 2{\cdot}135963.$$

If we had taken the gradient of the chord $\alpha\beta$, which is 14·4053, or say, 14·4, we should have calculated

$$x = 2{\cdot}13597.$$

Now it is evident that the true value lies between that given by the best tangent value and that given by the chord, so that the true value of x lies between 2·135963 and 2·135970. I have gone into the work in great detail in this case to emphasize the fact that when we have two points very near together, the curve and its chord are nearly coincident, and the chord may be substituted for the curve. This simplification is often of great use. Another point to notice is that we need not calculate the gradient to very many figures.

(2) Take the root near OQ.

When $x=4{\cdot}92,\quad f(x)=+0{\cdot}149888\,(\delta),$

and when $x=4{\cdot}91,\quad f(x)=-0{\cdot}071629\,(\gamma),$

and the gradient of the chord $\gamma\delta$ is 22·15. The gradient of the tangent at γ is 22·04, or say 22;

$$\therefore\ x \text{ is between } \quad 4{\cdot}913238 \quad \text{and} \quad 4{\cdot}913256.$$

The tangent is likely to give a better result than the chord, so we may put

$$x=4{\cdot}91325.$$

The diagram is left to the student.

(3) The negative root is about $-3{\cdot}05$.

When $x=-3{\cdot}05,\ f(x)=-0{\cdot}032625,$ and $f'(x)=41{\cdot}3$;

$$\begin{aligned}\therefore\ \text{the root} &= -3{\cdot}05+\frac{0{\cdot}032625}{41{\cdot}3}\\ &= -3{\cdot}05+0{\cdot}00079\\ &= -3{\cdot}04921.\end{aligned}$$

The three roots should add up to 4, which is the coefficient of $-x^2$ in the equation. We may therefore take them as

$$\left.\begin{array}{r} 2{\cdot}13596\\ 4{\cdot}91325\\ \text{and}\qquad -3{\cdot}04921\end{array}\right\}\text{the sum of which is 4.}$$

These roots have been calculated to excessive minuteness to illustrate the method. In practice much less accuracy is needed, and the roots would therefore be obtained more speedily and with less trouble. For instance, 2·136, 4·913, and $-3{\cdot}049$ would be ample accuracy in most cases.

In the work of approximation the diagram would show at once whether the correction was to be positive or negative, but it will be well to work out a formula which will enable us to work correctly, if desired, without a diagram.

Suppose that a is the first approximate value of x and that h is the correction, and that we have calculated $f(a)$ and $f'(a)$.

Then $$f(a+h)=0$$

and $$f(a+h)-f(a)=-f(a);$$

but $f(a+h)-f(a)$ is, when h is small, the differential of $f(a)$, and therefore $=hf'(a)$, if we retain only the first power of h. (h is the same thing as dx, being a small increment of x).

$$\therefore\ hf'(a)=-f(a);$$

$$\therefore\ h=-\frac{f(a)}{f'(a)};$$

$$\therefore\ \text{the root is } a-\frac{f(a)}{f'(a)}.$$

If we think of the formula from the geometrical point of view in connection with a diagram, if we denote the ordinate $f(a)$ by y, and the gradient $f'(a)$ by $\tan\phi$, the root is

$$a-y\cot\phi.$$

If we use the chord joining two points whose abscissae are a and b, the gradient of the chord is

$$\frac{f(b)-f(a)}{b-a},$$

and the approximate root is

$$a-\frac{b-a}{f(b)-f(a)}\times f(a)=\frac{af(b)-bf(a)}{f(b)-f(a)},$$

provided $f(a)$ and $f(b)$ are small and of opposite sign.

88. The two modes of approximating may be called the 'tangent' method and the 'chord' method. Both depend for their success on the change of direction not being very rapid near the crossing point.

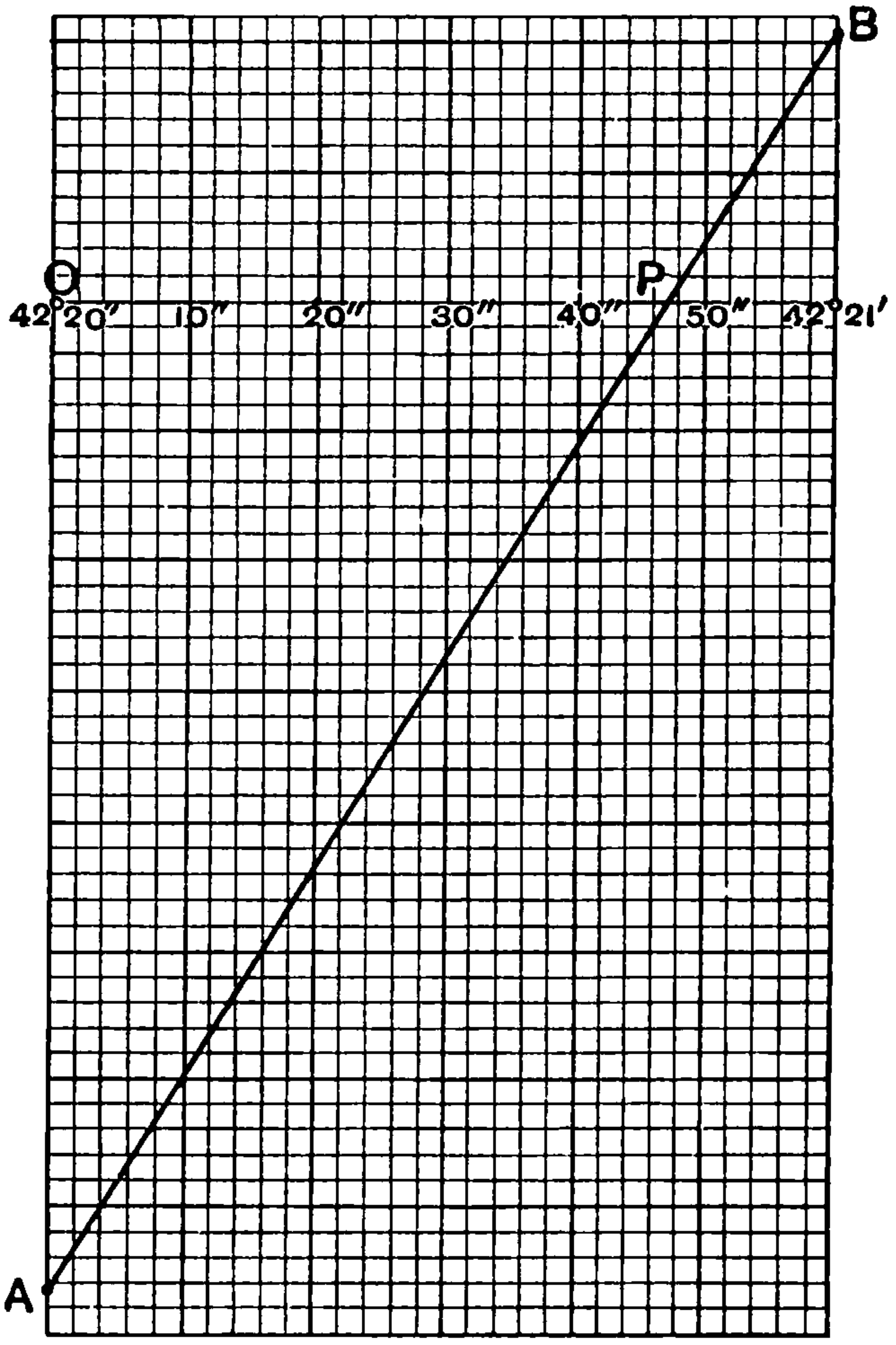
More accurate approximation formulae could be devised, taking the curvature of the curve into account, and their investigation is interesting; but if we make good use of the graphic method of approximation we can always obtain a first approximation to the root good enough to insure h being small, or the change of curvature between two points on either side of the crossing point being small. We may, therefore, be content with the methods already given.

Let us take another example. Find the angle whose circular measure is equal to its cosine.

Let x be the circular measure of the angle; the equation is $f(x) \equiv x - \cos x = 0$.

By help of the tables we can find two values of x so near together, on either side of the required value, that the chord of the curve $y = f(x)$ may be taken for the curve itself.

$$\begin{array}{rll}
 & \underline{42^\circ\ 20'} & \\
x = & \overline{0{\cdot}7388561} & \\
\cos x = & \underline{0{\cdot}7392394} & \\
x - \cos x = & -0{\cdot}0003833 & (\text{A})
\end{array}
\qquad
\begin{array}{rll}
 & \underline{42^\circ\ 21'} & \\
x = & \overline{0{\cdot}7391469} & \\
\cos x = & \underline{0{\cdot}7390435} & \\
x - \cos x = & +0{\cdot}0001034 & (\text{B})
\end{array}$$



The diagram shows that the required angle is about 42° 20′ 47″. Calculating OP from the figure, we have

$$\text{OP} = \frac{3833}{4867} \text{ of } 60''$$
$$= 47\tfrac{1}{4} \text{ seconds.}$$

89. Special graphic representation of the roots of quadratic equations. These equations can be so simply solved in surd form by the ordinary method that a graphic method is not of much use, and I only introduce the following method because of its extreme simplicity.

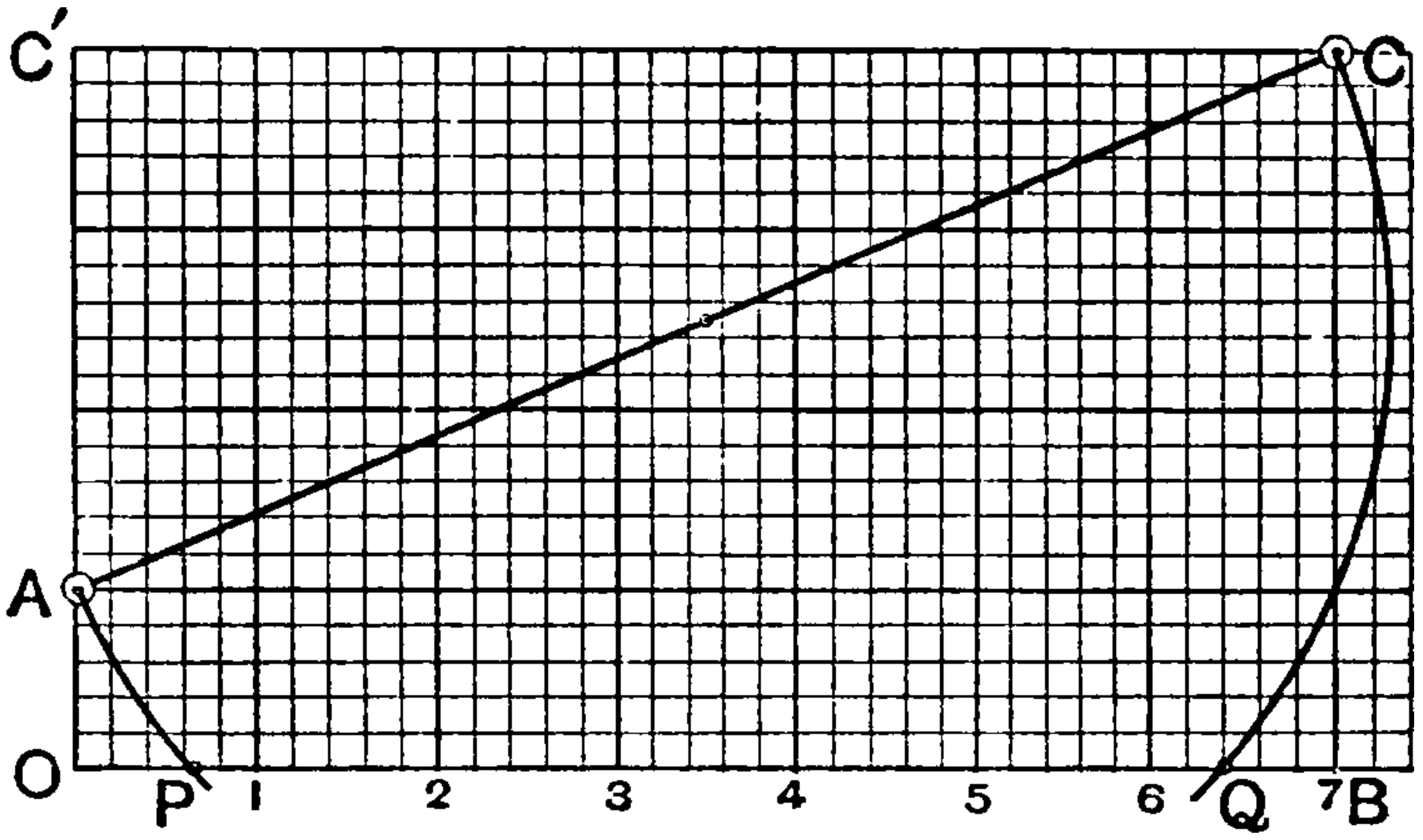
Let the equation be $x^2 - bx + c = 0$.

We could of course draw the curve $y = x^2 - bx + c$, and find where it cuts the x-axis. But the method now to be explained is simpler for securing the first approximation to the roots, and the general method can then be used for the final approximation to each root.

Draw, as shown in the diagram,

$$OA = 1, \quad OB = b, \quad \text{and} \quad BC = c,$$

the three coefficients being supposed positive in the figure. If b or c are negative they must be measured in opposite directions to those shown.



The angles at O and B are right angles.

Join AC, and on it as diameter describe a circle cutting OB in P and Q.

Then OP and OQ are the roots of the equation. This will be demonstrated if we can show that $OP + OQ = b$, and $OP \cdot OQ = c$.

Now, if OA cuts the circle again in C′, it is obvious that $OC' = BC = c$; also $OP = QB$;

$$\therefore \ (1) \ OP + OQ = OQ + QB = b,$$

$$(2) \ OP \cdot OQ = OA \cdot OC' = c, \ \text{since } OA = 1;$$

∴ OP and OQ are the roots. In the figure they are both positive, being measured to the right of O. If in any diagram either of them is to the left of O, that root is negative.

Example 1. Take, as an example, $x^2-7x+4=0$.
The roots are about $0{\cdot}6$ (+) and $6{\cdot}4$ (−). (See figure.)

For further approximation, if desired,

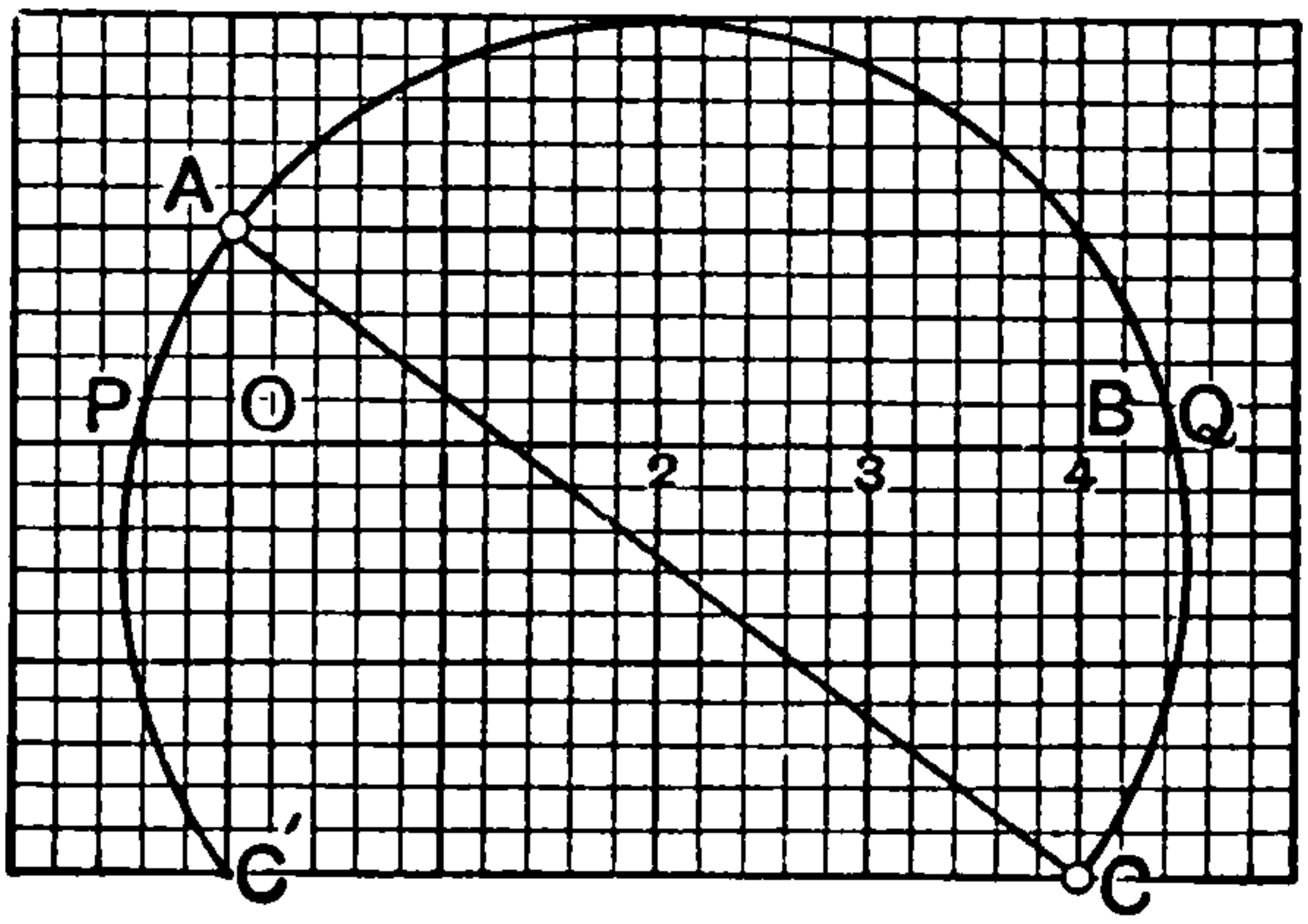
when $x=0{\cdot}6$, $f(x)=0{\cdot}16$,
$f'(x)=-5{\cdot}8$;
$\therefore\ x_1=0{\cdot}6276$;
and, when $x=6{\cdot}4$, $f(x)=0{\cdot}16$,
$f'(x)=+5{\cdot}8$;
$\therefore\ x_2=6{\cdot}3724$.

This makes $x_1+x_2=7$ as it should. The product is $3{\cdot}9993$ which is nearly the correct value 4.

If we are not satisfied with these values we can correct by the method suggested in Ex. 2, thus obtaining $x_1=0{\cdot}6277$ and $x_2=6{\cdot}3723$.

The student may usefully notice that the curve $y=x^2-bx+c$ (not shown in the diagram) goes through the points C, C′ in the figure, and has the gradients $\pm b$ at these points, so that the tangent at C is perpendicular to AB. The curve goes, of course, also through the points P and Q.

Example 2. Take, as another example, $x^2-4x-2=0$.
The approximate roots are $-0{\cdot}4$ and $+4{\cdot}4$. (See figure.)



When $x=-0{\cdot}4$, $f(x)=-0{\cdot}24$
and $f'(x)=-4{\cdot}8$;
$\therefore\ h=-\frac{1}{20}$;
and similarly in the case of $4{\cdot}4$, where $h=+\frac{1}{20}$;

$$\therefore \text{ the roots are } -0{\cdot}45 \text{ and } +4{\cdot}45.$$

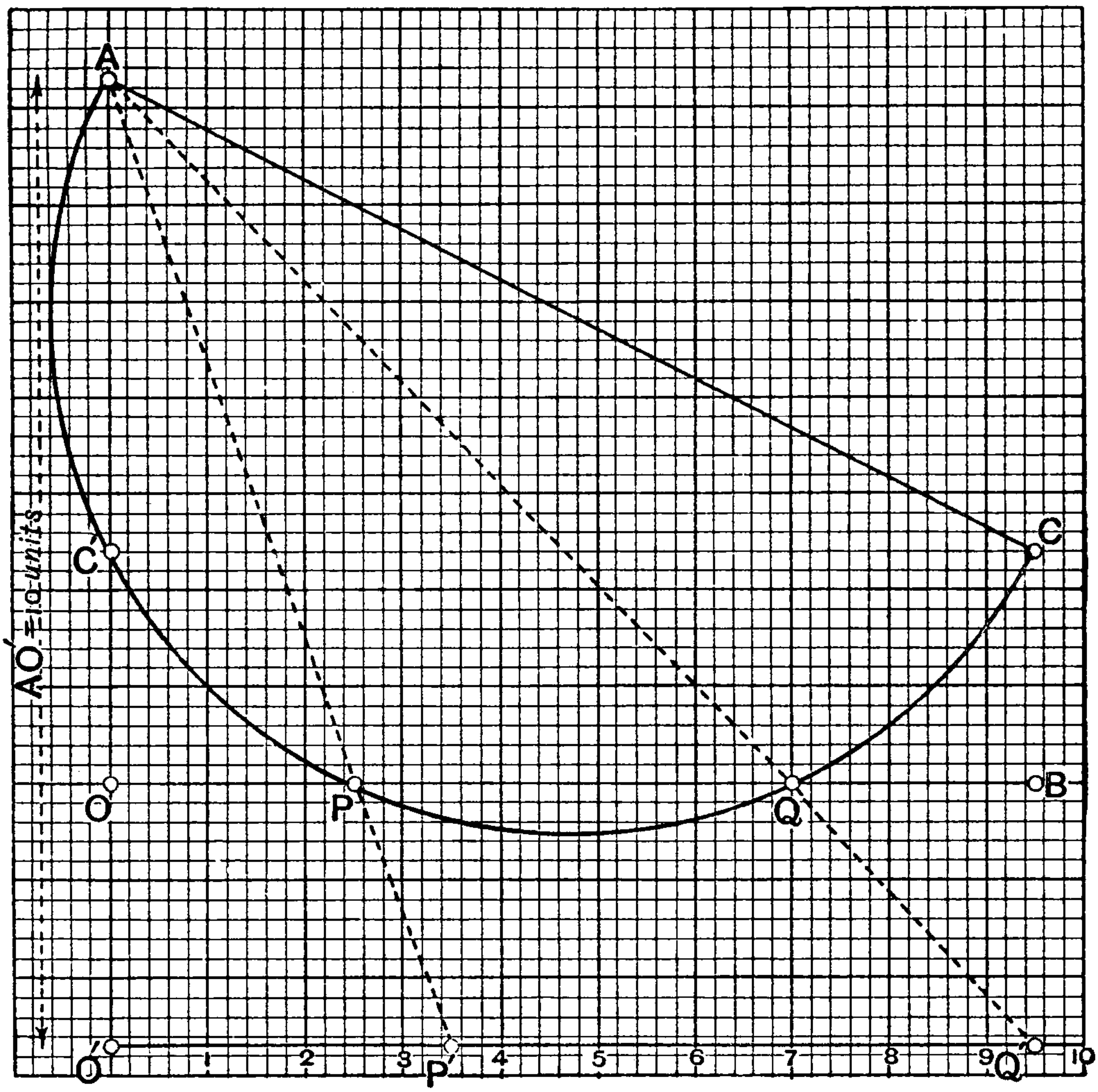
The sum of these roots is 4, as it should be since the second term in the equation is $-4x$. The easiest way to correct further is to note that the product of the roots ought to be -2, and that any error must be equally shared between the two. Thus $({\cdot}45+e)(4{\cdot}45+e)=2{\cdot}0025+4{\cdot}9e$, if we neglect the square of e; $\therefore\ e=-0{\cdot}0005$ to 4 decimal places, and the roots are $-0{\cdot}4495$ and $+4{\cdot}4495$.

90. If the equation is $ax^2-bx+c=0$, we may, instead of dividing by a, and proceeding as above, do as follows:

Draw $\left.\begin{aligned}OA&=a,\\ OB&=b,\\ BC&=c,\end{aligned}\right\}$ [See figure of example below]

then, drawing the circle on AC as diameter, cutting OB in P and Q, we shall find that OP and OQ are a times the roots.

For, if x_1 and x_2 are the roots, $x_1+x_2=\dfrac{b}{a}$, and $x_1x_2=\dfrac{c}{a}$; whereas $OP+OQ=b$, and $OP\,.\,OQ=ac$.



If, then, we draw a line O′P′Q′ parallel to OB, at unit distance from A, and take the points O′, P′, Q′ on it where it is cut by AO, AP, AQ respectively, the roots are O′P′ and O′Q′.

Or, of course, we may, if we please, measure OP and OQ and divide them numerically by a instead of graphically.

Or, if we do use the graphic method of reducing OP and OQ we may draw the parallel line O′P′Q′ at any convenient distance from A (*e.g.* 10 units, as in the figure), if we think unit distance too small or too great for convenience, and then reduce finally by easy division or multiplication as the case may be.

Example. $7{\cdot}3x^2-9{\cdot}5x+2{\cdot}4=0$.
The approximate roots are 0·35 and 0·95. (See figure.)
The corresponding values of $f(x)$ are $-0{\cdot}03075$ and $-0{\cdot}03675$,
and ,, ,, $f'(x)$ are $-4{\cdot}39$ and $+4{\cdot}37$.
∴ ,, h are $-0{\cdot}0070$ and $+0{\cdot}0084$.
∴ the roots are 0·3430 and 0·9584.

We can check this result by adding the roots together. The sum ought to be equal to $9{\cdot}5\div 7{\cdot}3$, and will be found correct to 4 decimal places.

EXAMPLES.

1. In Ex. 1, p. 92 (viz. $x^2-7x+4=0$), having taken the approximate roots in the first instance as being 0 6 (+) and 6·4 (−), so that their sum is 7 as it should be; correct these values by the condition that the errors must be equal (the roots being $0{\cdot}6+e$ and $6{\cdot}4-e$), and that the product of the roots must be 4; e^2 being considered negligible.

2. Find a root of the equation $x^3=1+x$.

3. Find the root of the equation $x^5=3+2x$ which lies between 1 and 2.

4. Solve the equation $\theta=\cot\theta$.

5. Solve the equation $x=10\log_{10}x$.

6. Solve the equation $\cos\theta=\tan\theta$.

7. If b is the known reciprocal of a given number a, show that the correction to be applied to b when a is slightly altered $=-\dfrac{b\,da}{a}$.

8. The reciprocal of 3476 is ·000287687. Find the reciprocal of 3476·38.

9. Solve the equation $x^2-10x+7=0$, graphically.

10. Solve the equation $x^2-10x-7=0$.

11. Solve the equation $2x^2+8x-7=0$.

12. Show that, in the construction of Art. 90, OP and OQ are the roots of the equation $x^2-bx+ac=0$; and show that Euc. II. 11 and 14 are examples of graphic solution by this method.

Show also that PAO and QAO are the angles given by the equation $a\tan^2\theta-b\tan\theta+c=0$.

CHAPTER VII.

MAXIMA AND MINIMA VALUES OF A FUNCTION OF ONE VARIABLE.

91. It is often desirable to know under what conditions some particular function of a given variable shall have a maximum or minimum value, and what that maximum or minimum value may be. For instance, in Examples 15 and 16, p. 16, the question naturally suggested itself, what must be the dimensions of the beams so that their strength should be greatest? We solved these problems graphically by, in each case, expressing the strength (or, rather, a quantity proportional to the strength) as a function of the breadth, and then drawing the graph of the function, and seeing from the figure for what value of the breadth x the function y was greatest. We also suggested a method by which this value of x could have been calculated, as it must be the value of x for which the gradient of the function would be zero. This method is of general application, and applies to minimum values also if the function has any. There are in theory some exceptions, but they seldom or never occur in practice. We will however, in the course of the following investigation, refer to these exceptions.

If, then, we want to find the maximum or minimum values of any variable quantity, the first thing to do is to express the value of such quantity (which for definiteness we will call y) in terms of some convenient variable (which we will call x) on which its value depends; *i.e.* the first thing to do is to form the equation

$$y=f(x),$$

which expresses y in terms of x.

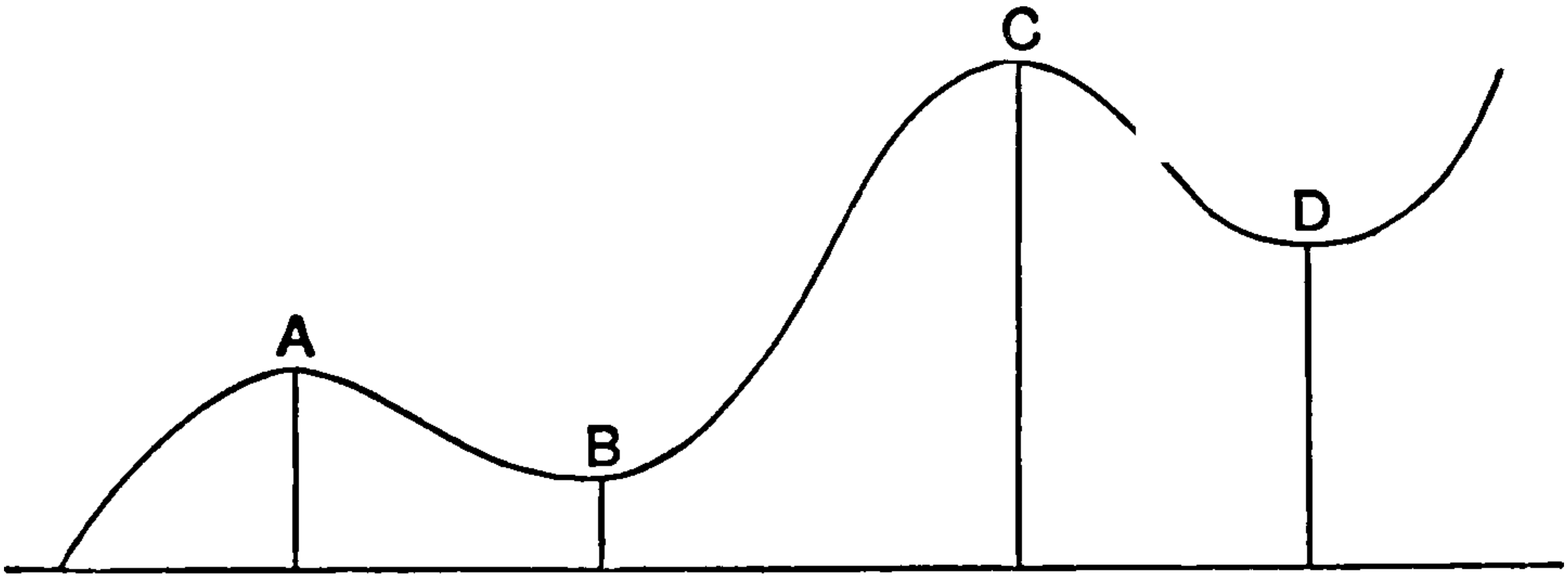
We have then to find for what value or values of x the function y will have a maximum or minimum value. For this purpose we will carefully define what we mean by such values.

92. Definition of maximum and minimum.

A maximum value of y is a value which is greater than adjacent values, however near; or, in symbols, if $y=f(x)$, and $x=a$ makes y a maximum, *i.e.* if $f(a)$ is a maximum value of y, $f(a)$ will be greater than either $f(a+h)$ or $f(a-h)$, however small h may be. (Note—not however *great* h may be, but however *small*.)

A minimum value of y is a value which is less than adjacent values, however near; or, in symbols, if $x=b$ makes $f(x)$ a minimum, $f(b)$ will be less than either $f(b+h)$ or $f(b-h)$, however small h may be.

93. A function may have several maximum and several minimum values, as, for example, in walking over an undulating road we may go over several hills and through several valleys. On the top of each hill the height above sea-level is a maximum, and at the bottom of each valley the height above sea-level is a minimum. Each maximum is greater than the adjacent minima, but not necessarily greater than all minima.



Thus, in the diagram, the ordinates at A and C are maximum ordinates, and those at B and D are minimum ordinates, and yet the ordinate at A is less than the ordinate at D.

In a continuous function maxima and minima values evidently occur alternately.

94. Increasing and decreasing functions.

If, as x increases, y also increases, y is called an increasing function of x. It is evident that its 'gradient' $\frac{dy}{dx}$ is positive.

If, on the contrary, as x increases, y decreases, y is called a decreasing function. It is evident that in this case $\frac{dy}{dx}$ is negative.

Most functions are increasing functions for some values of x and decreasing functions for other values. For instance, in Example 15, p. 16, y is an increasing function from $x=0$ to $x=3\frac{1}{3}$, and it then becomes a decreasing function; or, in other words, its gradient is positive between $x=0$ and $3\frac{1}{3}$, but is negative for greater values of x. When $x=3\frac{1}{3}$ the gradient of this curve is zero, and the ordinate is a maximum.

95. General condition that $x=a$ shall make $f(x)$ a maximum.

If, as x increases from some smaller value to the critical value a, the function $f(x)$ continually increases, *i.e.* has its gradient $(f'(x))$ positive; but when x is greater than a, $f(x)$ begins to decrease, *i.e.* $f'(x)$ becomes negative: $f(a)$ is a maximum value of $f(x)$.

General condition that $x=a$ shall make $f(x)$ a minimum.

If, as x increases up to a, the function $f(x)$ continually decreases, *i.e.* $f'(x)$ is negative, and when x is greater than a, $f(x)$ begins to increase, *i.e.* $f'(x)$ becomes positive, $f(a)$ is a minimum value of $f(x)$.

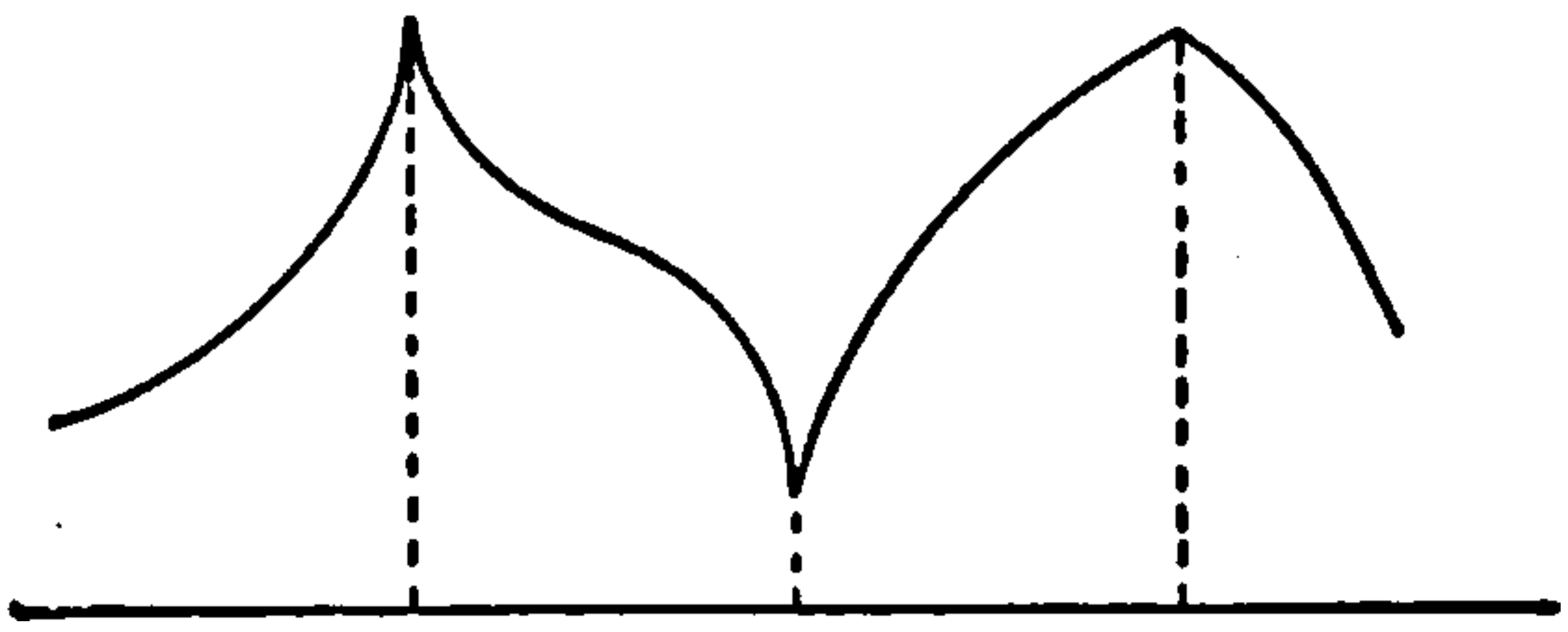
In each case the necessity is that $f'(x)$ shall change sign as x passes through the critical value. The distinction between the two cases is in the order of the change. Briefly

$f(a)$ is a maximum value of $f(x)$ if, as x increases through the value a, $f'(x)$ changes from $+$ to $-$; and

$f(a)$ is a minimum value of $f(x)$ if, as x increases through the value a, $f'(x)$ changes from $-$ to $+$.

96. Now it is evident that, if $f'(x)$ changes gradually and does not become infinite, it cannot change sign except by passing through the value 0. Hence, in general, the primary condition for $f(a)$ to be a maximum or a minimum value of $f(x)$ is that $f'(a)$ shall equal 0.

The exception occurs when $f'(a)$ is infinite, or when there is an abrupt change of gradient. In practice the latter case never occurs, and the former very seldom. The shape of the curve in such cases is shown in the diagram.

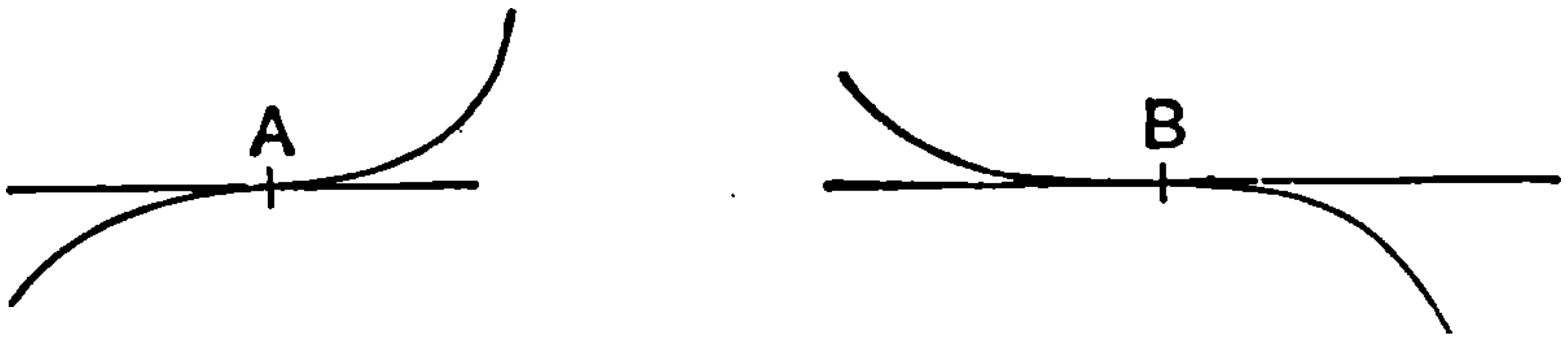


97. Points of inflexion.

If $f'(a)=0$, but the gradient on either side has the same sign, there is a peculiarity in the curve at the point, but $f(a)$ is neither a maximum nor a minimum value of $f(x)$.

Such cases are shown in the adjoining figures.

Such a point on the curve is called a point of inflexion.



Gradient + on both sides of A. Gradient – on both sides of B.

Points of inflexion may occur when the tangent is not horizontal, but they have no connection with the present investigation. The general definition of a point of inflection is that it is a point at which the curve crosses its tangent.

98. From what we have seen it is evident that the first thing to do when we want to find the maximum and minimum values of any function $f(x)$ is to form the auxiliary function $f'(x)$, and to find what values of x make $f'(x)=0$, *i.e.* to solve the equation $f'(x)=0$. This may be done either graphically or by calculation (see last chapter). The next thing is to see for each value of x so obtained—called the critical values of x, because they have to be critically examined—whether there is any change in the sign of $f'(x)$ as x increases through the critical value in question, and, if so, whether the change in $f'(x)$ is from $+$ to $-$, or from $-$ to $+$. If $f'(x)$ can become infinite for any finite value of x we ought also to see whether there is a change of sign in $f'(x)$ on either side of this value of x.

In many practical examples, however, it will be obvious whether the function is a maximum or a minimum without the analytical method of discovering which it is. In such cases we merely put $f'(x)=0$, and do not investigate the change in its sign on either side. The student can work the examples 1–9 at the end of this chapter in this simple manner. It might be well for him to solve some of these now, before reading the rest of the chapter.

[CAUTION.—Beginners, from want of careful thought, often seem to imagine that it is x or $f'(x)$ which becomes a maximum or minimum in the above cases. It is not so: it is $f(x)$ which has a maximum or a minimum value when the above conditions are satisfied. The beginner, too, often finds general statements difficult to follow; he should try and have some definite idea in his mind; probably the best general notion is that furnished by the varying height above sea-level of a man walking along an undulating road, as that most readily lends itself to the conception of several maxima and minima. Other instances, generally of single maximum or minimum values, will gradually accumulate in his mind as he works at the examples at the end of this chapter. He should work at some of them, and then read the above general statements again and again, till they become self-evident.]

99. The following example will illustrate the method:

Find the maximum and minimum values of

$$2x^3-9x^2+12x.$$

Denoting this by $f(x)$, our auxiliary function will be

$$f'(x)=6(x^2-3x+2)=6(x-1)(x-2).$$

Hence the critical values of x are 1 and 2, since these values make $f'(x)=0$. The neatest way of recording the result of examining the sign of $f'(x)$ for different values of x will be to make a short table as follows:

x	$f'(x)$	$f(x)$
less than 1 (*e.g.* 0)	+	increases with x
1	0	5 (*max. value*)
between 1 and 2	−	decreases as x increases
2	0	4 (*min. value*)
greater than 2	+	increases with x

The statements in italics in the above table are made last. What we first establish (by inspection) is that $f'(x)$ is positive when x is less than 1, that it becomes 0 when $x=1$, that then $f'(x)$ becomes negative and remains negative until $x=2$ when it becomes 0 again, afterwards becoming positive and remaining so however great x may be. From these fluctuations in the auxiliary function $f'(x)$ we deduce that as x gradually increases from any small value (say, $-\infty$), $f(x)$ is an increasing function up to $x=1$, then $f(x)$ is a decreasing function till $x=2$, after which it becomes and remains an increasing function. Therefore, finally, $f(x)$ has its maximum value when $x=1$ and its minimum value when $x=2$. Then by calculation we find that the maximum value is 5, and the minimum value is 4.

The student should draw the graph of $f(x)$, and also, on the same scale as and immediately under the former diagram, he should plot the graph of $f'(x)$, preferably by a dotted curve to show that it is merely an auxiliary function. If he then compares the graphs carefully with the above table he will see that all the statements made in it are visibly true. He will also notice that in this example the maximum value is not greater than all other values, but only greater than adjacent values on either side; and similarly the minimum value is not less than all other values, but only less than adjacent ones.

100. In the above example the roots of $f'(x)=0$ are whole numbers. Of course in practice this is not necessarily or even generally the case. The roots can then be obtained by some general method such as those given by last chapter. The graphic method is an excellent one, as very often the values obtained from the diagram will be sufficiently accurate for all practical purposes, since in the neighbourhood of a maximum or a minimum value of $f(x)$ a small error in the determination of the critical value of x makes hardly any appreciable difference in $f(x)$. Indeed, so much is this the case, that in many cases it might seem better not to worry about theory at all, but merely plot $f(x)$ itself and read off its maximum and minimum values. This method in fact was adopted in the cases of strengths of beams in Examples 17, 18, p. 16.

The counter advantages of using the auxiliary function $f'(x)$ are (1) that it is often easier to draw the graph of $f'(x)$, being of lower degree, and (2) that we can generally determine the critical values of x more accurately from this function. *Graphically*, it is easier to determine accurately what values of x make $f'(x)=0$ than it is to determine accurately what values make $f(x)$ a maximum, and, as either method gives the values of x required, it is at any rate well to be able to use the more accurate method even when we intend to work graphically. When we work by *calculation*, we *must* use the auxiliary equation $f'(x)=0$, there is no way of avoiding it. Indeed it is often so quick and simple to use that it hardly needs this defence.

101. We will take as our second problem a geometrical one.

Find the maximum cone inscribable in a given sphere.

We have now first to find our function $f(x)$, *i.e.* we must express the volume of any inscribed cone in terms of some convenient variable x (though of course we need not actually use the symbol x unless we like).

Take as variable the height, h, of the cone.

The axis AN $(=h)$ of the cone will obviously pass through the centre O of the sphere. Let NP be the radius of the base of the cone, so that the area of the base $=\pi \, . \, \text{NP}^2$. The volume is $\frac{1}{3}$ base × height.

Let a be the radius of the sphere, so that $\text{OA}=\text{OP}=a$.

Then $\text{NP}^2=\text{OP}^2-\text{ON}^2=a^2-(h-a)^2=2ah-h^2$;

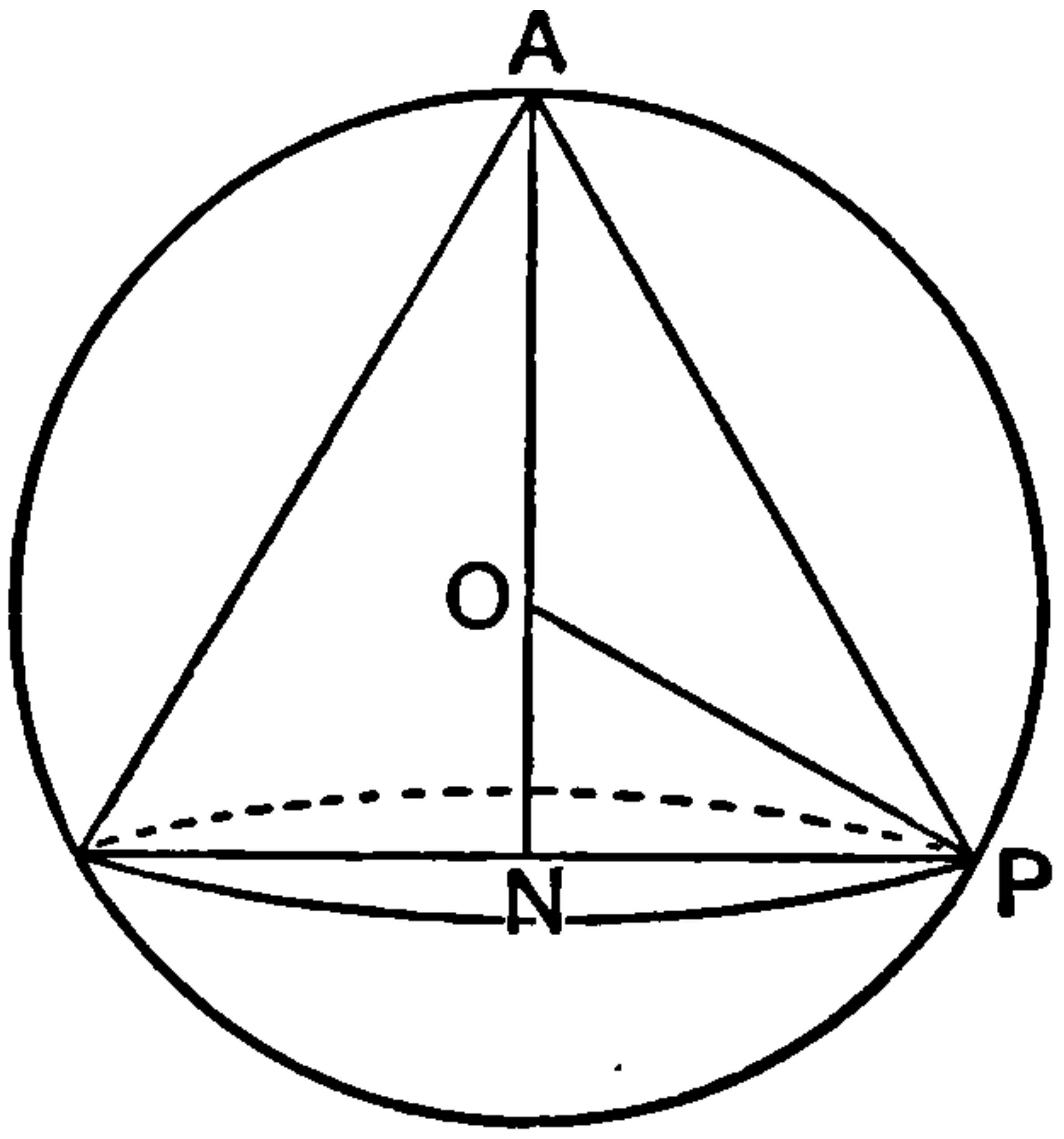
$$\therefore \text{ the volume of the cone} = \tfrac{1}{3} \, . \, \pi(2ah-h^2) \, . \, h$$
$$= \tfrac{1}{3}\pi(2ah^2-h^3).$$

We have, therefore, to make $2ah^2-h^3=$ a maximum.

Hence its differential coefficient, viz., $4ah - 3h^2$, must $= 0$;

$$\therefore\ h = 0 \text{ or } \frac{4a}{3}.$$

One of these must make the volume a maximum, and the other must make it a minimum.



Now in a problem of this kind we need no theory to distinguish between maximum and minimum values. It is obvious that $h = 0$ makes the volume of the cone $= 0$, which must therefore be the minimum volume; hence $h = \frac{4a}{3}$ makes the volume a maximum.

Putting this value in the expression $\frac{1}{3}\pi \cdot h^2(2a - h)$ we find the maximum volume $= \frac{1}{3}\pi \cdot \frac{16a^2}{9} \cdot \frac{2a}{3} = \frac{32\pi a^3}{81}$ which is $\frac{8}{27}$ of the volume of the sphere (viz. $\frac{4}{3}\pi a^3$).

If we want to make a graph of the volume, the neatest way is to take as our function to be plotted the ratio of the conical volume to that of the sphere, as that gets rid of the trouble of the factor $\frac{1}{3}\pi$.

This ratio is $\frac{2ah^2 - h^3}{4a^3} = \frac{2h^2}{4a^2} - \frac{h^3}{4a^3}$.

If lastly we take as x the ratio of h to the diameter $2a$ of the sphere, the function reduces to

$$2x^2 - 2x^3 = 2x^2(1 - x),$$

which is easy to graph. We have to draw it between $x = 0$ and $x = 1$.

We know, moreover, that the function is a maximum when $x = \frac{2}{3}$. We may therefore plot the points corresponding to

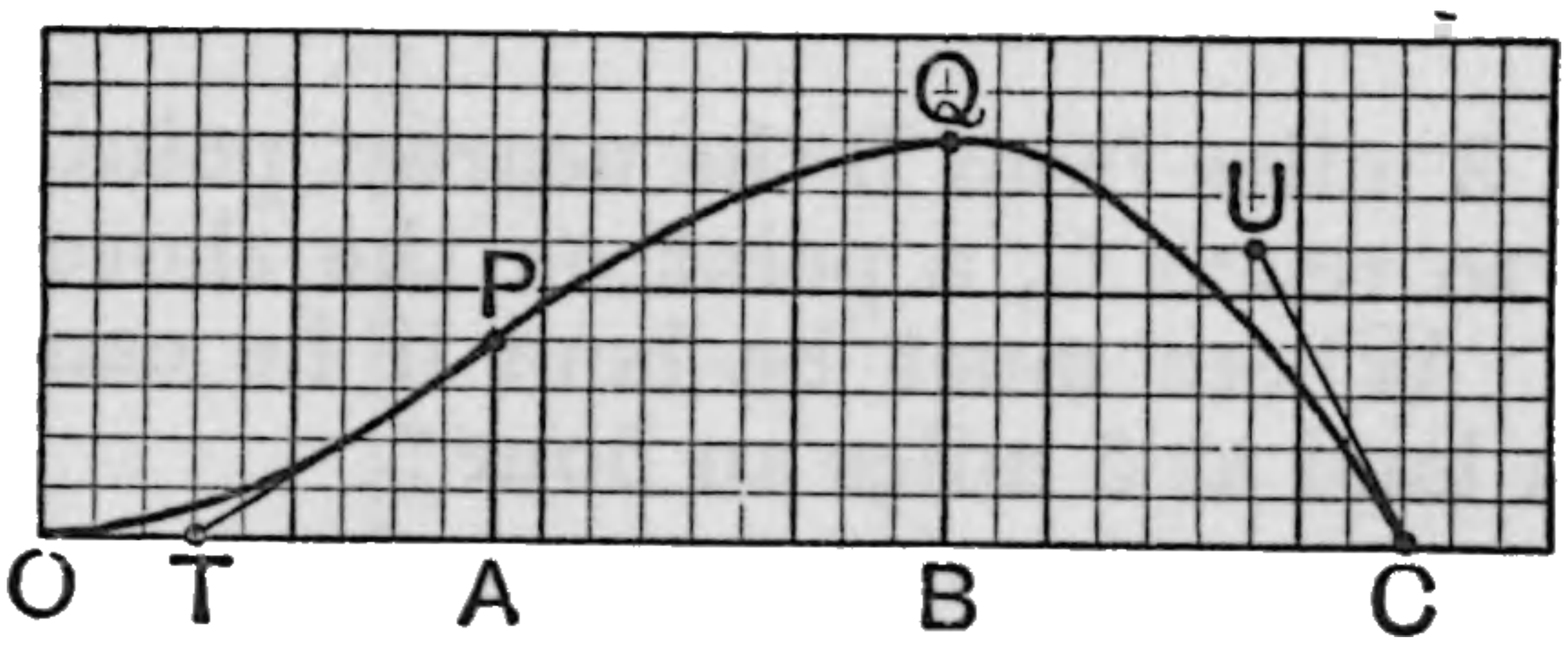
$$x = 0,\ \tfrac{1}{3},\ \tfrac{2}{3},\ 1,$$

and these, together with the gradients at the same points, will be amply sufficient for a good diagram.

The gradient is given by $4x - 6x^2$.

The table is

x	$f(x)$	$f'(x)$	Points.
0	0	0	O
$\frac{1}{3}$	$\frac{4}{27}$	$\frac{2}{3}$	P
$\frac{2}{3}$	$\frac{8}{27}$ (max.)	0	Q
1	0	-2	C



If we take each small division on the squared paper $=\frac{1}{27}$, for both horizontal and vertical measurements, the figure will be as shown. The tangent PT at P will cut the axis of x in T, such that $TA=6$ divisions, because then its gradient

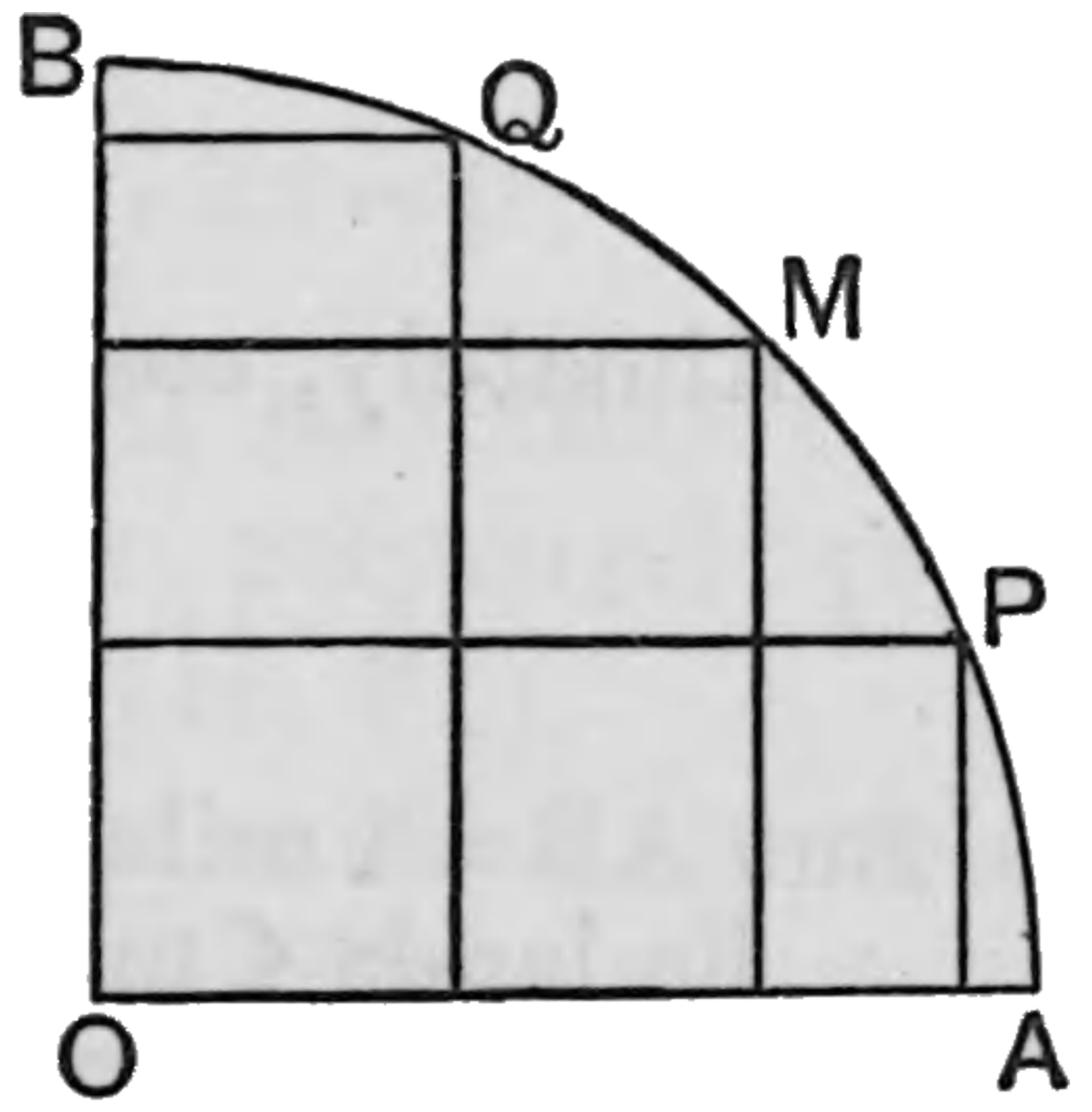
$$\frac{AP}{TA}=\frac{4}{6}=\frac{2}{3}.$$

The tangent at C is shown as CU, with a gradient $=-2$.

Of course we might have reduced the above fractions to decimals and plotted decimally.

102. The following considerations will often enable us to avoid the calculus altogether.

Axiom. Between two equal values of a function a maximum or minimum value of the function must occur, and will ultimately coincide with them when they are infinitely near together.



Example. Find the maximum rectangle which can be inscribed in the quadrant AOB of a circle.

Take the rectangles, as shown in the figure, whose angular points P, Q are symmetrically situated in the arc AB. Then evidently the areas of the two rectangles are equal.

When P and Q are infinitely near together they ultimately meet at the middle point M of the arc.

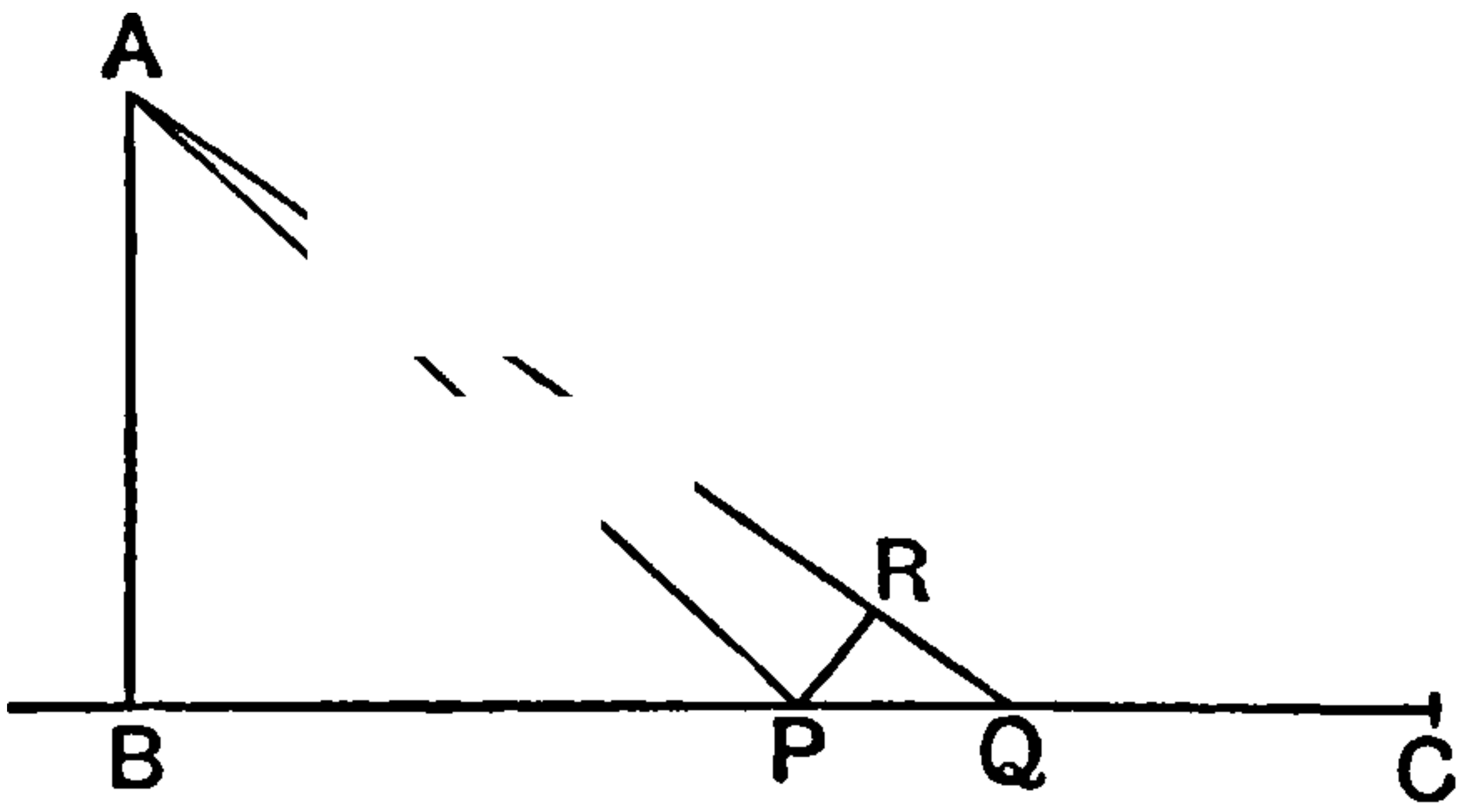
Hence M is the angular point of the maximum rectangle, which is therefore a square.

This answer is really obvious, for M is in a position of symmetry, and a position of symmetry will always make the function a maximum or a minimum.

The following are examples of symmetry: A loop of string lying in a plane surface will enclose the greatest area when it is circular. The solid of greatest volume for a given surface is the sphere.

103. As another example of the axiom take the following:

A boatman at A, 3 miles from B the nearest point of the shore, which is straight, wishes to reach C, a point on the shore 6 miles from B, in the shortest time. Where must he land if he can row at 4 miles an hour, and travel on land at 5 miles an hour?



Let APC and AQC be two paths infinitely close together which will take equal times. They will be on either side of the required path and will ultimately coincide with it.

Cut off AR=AP. Then PR is ultimately perpendicular to both AP and AR.

Now by the APC path, he travels the extra land piece PQ,
and by the AQC ,, ,, ,, sea piece RQ.

These must take equal times. If these distances are measured in fractions of a mile, the times are $\frac{PQ}{5}$ and $\frac{RQ}{4}$ hours;

$$\therefore \frac{PQ}{5}=\frac{RQ}{4},$$

i.e. ultimately, $\cos Q=\frac{4}{5}$ [when P and Q ultimately coincide],

$$\therefore \tan Q=\tfrac{3}{4}, \quad \textit{i.e.} \ \frac{AB}{BQ}=\tfrac{3}{4}.$$

But AB=3 miles; ∴ BQ=4 miles;
∴ he lands 4 miles from B, *i.e.* 2 miles from C,
i.e. he rows 5 miles, and walks 2, and he takes $\frac{5}{4}+\frac{2}{5}$ hours=1 hour 39 minutes, not counting the time occupied in landing, which would be the same in any case.

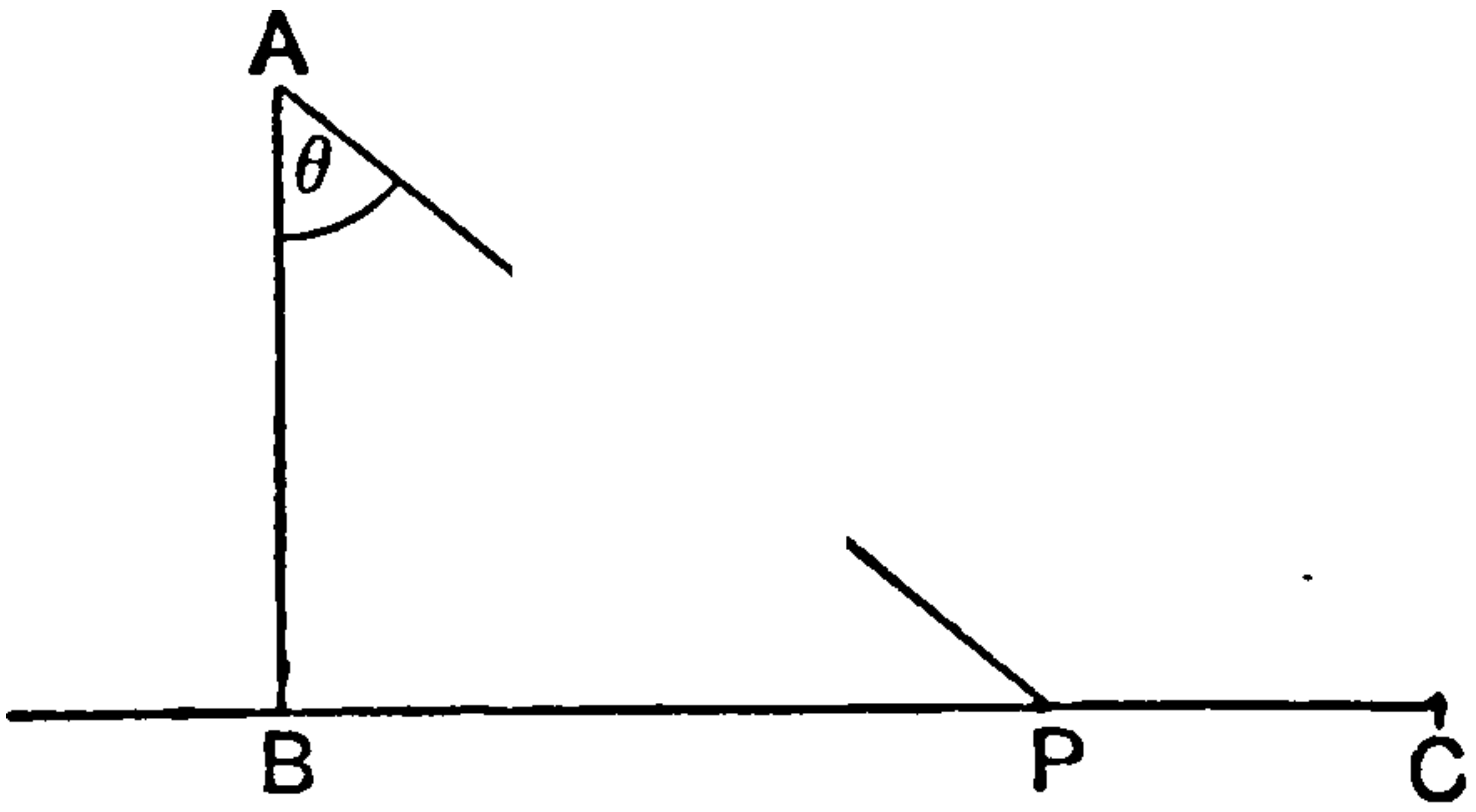
If we do the same problem by the calculus method the neatest way is to express the time in terms of the angle θ which the path AP makes with AB.

$$AP = AB \sec\theta = 3\sec\theta, \text{ in miles,}$$

$$PC = BC - BP = 6 - 3\tan\theta, \text{ in miles;}$$

$$\therefore \text{ time, in hours, } = \frac{3\sec\theta}{4} + \frac{6 - 3\tan\theta}{5}$$

$$= \frac{3}{20}(8 + 5\sec\theta - 4\tan\theta).$$



The differential coefficient of this function has to be zero;

$$\therefore \ 5\sec\theta\tan\theta - 4\sec^2\theta = 0;$$

$$\therefore \ \sin\theta = \tfrac{4}{5}$$

This leads to the same answer as before.

104. Maximum and minimum gradient. Point of inflexion. A question which often arises in connection with the graph of a function is as to where the *gradient* is a maximum or minimum. Thus, find the point on the curve $y = 2x^2 - 2x^3$ where the gradient is a maximum, *i.e.* where the curve is steepest.

The gradient is given by the equation $\frac{dy}{dx} = 4x - 6x^2$, or let us say, for brevity, $y_1 = 4x - 6x^2$. We have therefore to make y_1 a maximum. Hence $\frac{dy_1}{dx}$ must $= 0$, *i.e.* $4 - 12x = 0$; $\therefore x = \frac{1}{3}$. If we look at the diagram (p. 103) we shall see that this is evidently the case, the point P being the required point. Such a point is always a point of inflexion (Art. 97).

There is no difficulty in connection with such a problem except as to the meaning of maximum or minimum when the gradient is negative.

In the case of a negative gradient, the point where the curve is steepest is technically a point of minimum gradient, on the principle that a large negative quantity is algebraically less than a small one. If we like we may call it a negative maximum. The student might notice also the gradients at A and B in the diagram on p. 98. The gradient at each of these points is zero, but at A it is a minimum, because the adjacent gradients are positive, and at B it is a maximum because the adjacent gradients are negative. Of course a zero gradient is in general neither a maximum nor a minimum gradient, as usually the gradient is positive on one side and negative on the other. At such points it is the ordinate which is a maximum or a minimum, not the gradient. The gradient is never a maximum or minimum except at a point of inflexion, and at such a point it always is a maximum or minimum.

105. Occasionally it may be convenient to express the function, whose maximum and minimum values are required, in terms of two variables instead of one only, there being some given relation between the variables. The method of proceeding in such a case is best illustrated by an example.

Take the example No. 15, p. 16, where we have a rectangular beam of maximum strength to be cut out of a cylinder of 12 inches diameter. The equations are

$$bh^2 = \text{max.} \quad \text{......(1)}$$

and

$$b^2 + h^2 = 144, \quad \text{......(2)}$$

using h to denote the depth or height of the beam, so as not to confuse it with the sign d for differential.

Differentiating (1) gives us

$$h^2 db + 2bh\,dh = 0, \quad \text{......(3)}$$

and differentiating (2) gives

$$2b\,db + 2h\,dh = 0. \quad \text{......(4)}$$

These two equations between db and dh must be consistent, *i.e.* they must really be the same equation in different form. Hence the ratio between the coefficients of db must equal the ratio between the coefficients of dh,

$$i.e.\ \frac{h^2}{2b} = \frac{2bh}{2h},$$

$$i.e.\ \frac{h^2}{2b} = b;$$

$$\therefore\ h^2 = 2b^2.$$

Or, we might have found $\frac{db}{dh}$ from (3), and also from (4), and equated them, leading to the same result.

106. Another mode of discriminating between maximum and minimum values of y.

We have seen that y is an increasing function when the gradient $\left(\frac{dy}{dx} \text{ or } y_1\right)$ is positive, and a decreasing function when y_1 is negative.

Similarly the gradient is itself an increasing function when *its* differential coefficient $\frac{dy_1}{dx}$ is positive, and a decreasing function when $\frac{dy_1}{dx}$ is negative. This property can be made use of to facilitate the discrimination between the maximum and minimum values of y. Denote $\frac{dy_1}{dx}$ by the symbol y_2 for brevity.

Now when x increases through a value that makes y a maximum, the value of y_1 goes through the series of changes $+$, 0, $-$. Therefore y_1 is a decreasing function, and y_2 must be negative.

And when x increases through a value that makes y a minimum, the value of y_1 goes through the series of changes $-$, 0, $+$. Therefore in this case y_1 is an increasing function, and y_2 must be positive.

Hence, in general,

y is a maximum when $y_1 = 0$ and y_2 is negative,

and y is a minimum when $y_1 = 0$ and y_2 is positive.

Therefore, in any given example, after finding the values of x that make $y_1 = 0$ we may, if we like, see in each case whether y_2 is negative or positive, instead of going through the fundamental method of seeing whether the change of y_1 is from $+$ to $-$ or from $-$ to $+$. Of course if y_2 happens to be zero at any of these points, this new method fails, and we must fall back on the fundamental method. It is probable in such case that we have a maximum or minimum *gradient*, (*i.e.* a point of inflexion) and not a maximum or minimum ordinate. Examination of the change of gradient by the fundamental method will tell us which of all of them is the case. However, in *practical* cases, points where y_1 and y_2 are both zero seldom occur, and the examination of the sign of y_2 is a neat and convenient mode of discriminating.

Take, as example, $y=2x^2-2x^3$.

Then $y_1=4x-6x^2$, $=0$ when $x=0$ or $\frac{2}{3}$; and $y_2=4-12x$, which is positive when $x=0$, and negative when $x=\frac{2}{3}$.

Hence $x=0$ makes y a minimum, and $x=\frac{2}{3}$ makes y a maximum.

107. Further investigation of the case when $y_1=0$ and also $y_2=0$.

We have seen that

y is a maximum when $y_1=0$ and y_2 is negative,

and y is a minimum when $y_1=0$ and y_2 is positive.

Similarly if we want to discriminate between maximum and minimum gradients we may form the third auxiliary function $\frac{dy_2}{dx}$ or y_3, often called the third derived function, or third differential coefficient. We then see that

y_1 is a maximum when $y_2=0$ and y_3 is negative.

and y_1 is a minimum when $y_2=0$ and y_3 is positive.

The method will fail, *i.e.* the result is doubtful, if $y_3=0$.

From this, if we remember that when y is a maximum or minimum, y_1 is not, and when y_1 is a maximum or minimum y is not, it follows that y is not a maximum or minimum if $y_1=0$, $y_2=0$, and y_3 is not zero, but it *may* be if $y_1=0$, $y_2=0$, $y_3=0$.

Similarly y_1 is not a maximum or minimum if $y_2=0$, $y_3=0$, and y_4 is not zero. Hence in this case y *is* a maximum or minimum (provided, of course, $y_1=0$).

However, the reasoning is rather difficult to follow at this stage. It will be much easier when we have learnt a theorem called Taylor's Theorem (ch. XI.), and we shall see then that y is a maximum when $y_1=0$, $y_2=0$, $y_3=0$, and y_4 is negative, and a minimum when $y_1=y_2=y_3=0$ and y_4 is positive. But the theory is not very useful, as it is far better in such cases to fall back on the fundamental method of examining the sign of y_1 on each side of the critical point.

EXAMPLES.

1. Find the greatest cylinder that can be cut out of a conical block, the height of the cone being 3 feet and the diameter of its base being 2 feet.

2. Find the relation between the height (h) and radius (r) of the base of a cylindrical vessel open at the top (such for example as a bushel measure) when the internal surface is a minimum for a given capacity.

3. If the cylinder is closed at both ends (like a tin of preserved meat), find the ratio of h to r when the surface is a minimum for a given capacity (*i.e.* when it is made most economically).

4. Find the ratio of the height to the radius of the base of a conical vessel fulfilling the same conditions, (1) when it is open; (2) when it is closed by a flat circular plate.

5. Find the maximum cone of given slant height.

6. Find the maximum cylinder inscribable in a given sphere.

7. An open tank is to have a square base and vertical sides, so as to have a given capacity. Find the proportion of its depth to its width so that the expense of lining it with lead may be a minimum.

8. Find the shortest ladder which will reach from the ground to the wall of a house over an obstruction 6 feet high and 5 feet away from the wall. Find the angle of slope of the ladder.

9. Find the maximum and minimum ordinates of the curve

$$y=x^3-6x^2+12,$$

and also find the points of maximum gradient.

10. Find the maximum and minimum values of

$$3x^5-25x^3+60x.$$

11. Find the minimum value of $ax+bx^{-1}$, a and b being positive quantities. Show that it occurs when ax is equal to bx^{-1}.

12. Find the points of inflexion of the curve $x^2y=3(x-y)$.

13. Find the maximum and minimum values of $(x+1)(x-2)^2$.
Draw a graph of this function between $x=-2$ and $+3$.
Draw also a graph of its first derived function.

14. Find the maximum and minimum values of $(x+1)^2(x-2)^3$. Draw the graph.

15. Find the maximum and minimum values of $(x+1)^3(x-2)^4$. Draw the graph.

16. Find the maximum and minimum values of $\dfrac{x^2-x-1}{x-1}$. Draw the graph.

17. Find the maximum and minimum values of $2x^3-3x^2-36x+10$, and plot a graph of the function. Plot also a graph of its differential coefficient, and show the connections between the two curves.

18. Show that the maximum value of $a\cos\theta+b\sin\theta$ is $\sqrt{(a^2+b^2)}$.

19. Show that the minimum value of $a\sec\theta-b\tan\theta$ is $\sqrt{(a^2-b^2)}$.

CHAPTER VIII.

CURVATURE OF PLANE CURVES.

108. Definition.

If two points A, B are taken close together on a curve, the infinitely short arc AB is generally denoted by ds. If AT and BT are the tangents at A and B respectively, the angle between them is $d\phi$, if ϕ is the gradient-angle of AT, and $\phi + d\phi$ the gradient-angle of BT.

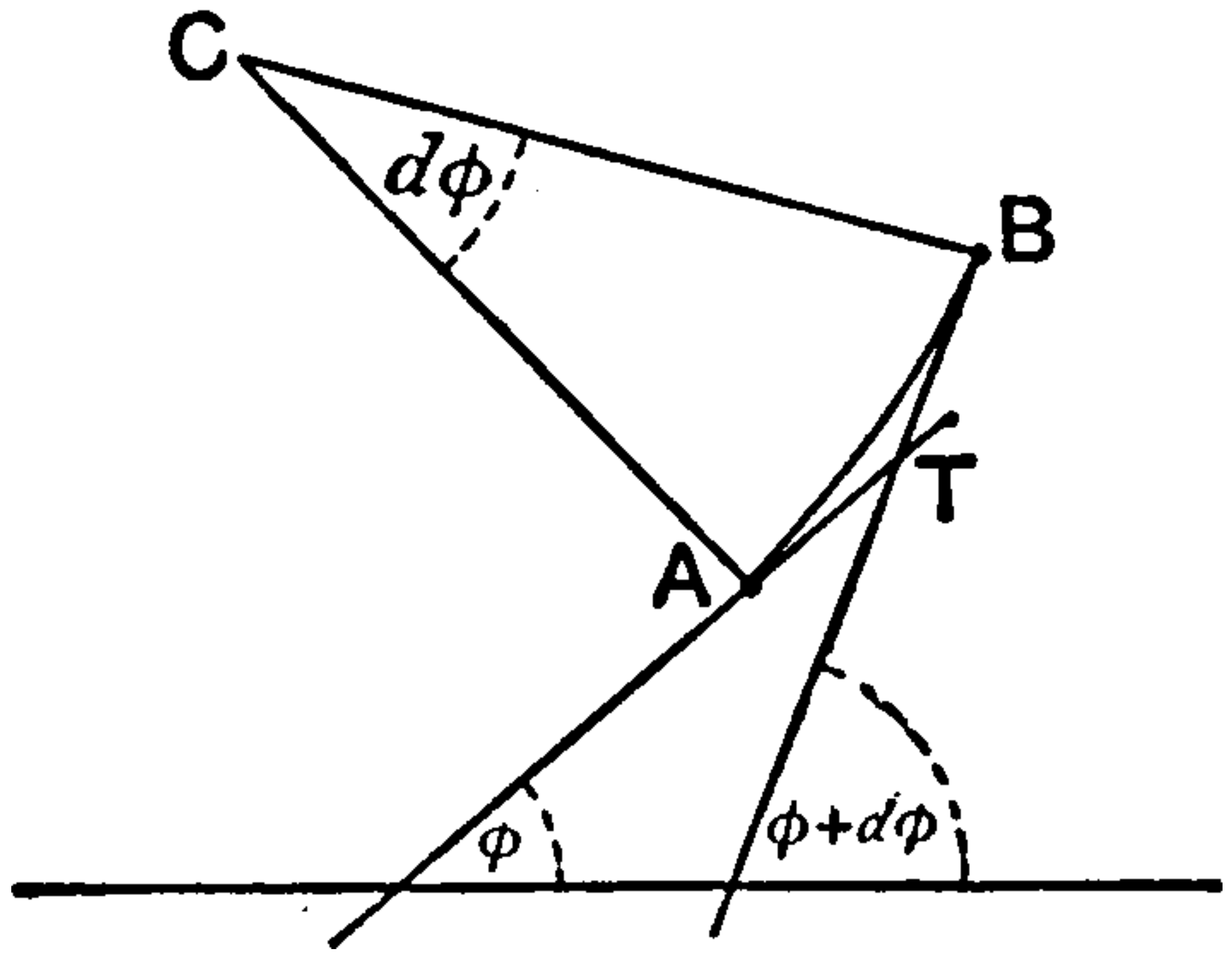
Then the curvature of the curve at A is defined to be the limiting value of the ratio $\frac{d\phi}{ds}$ when B is ultimately made to coincide with A, *i.e.* the curvature equals the ratio of the change of direction between A and B to the length of the arc AB, when AB is infinitely small.

If A and B are not infinitely close, this ratio may be called the *average* curvature of the part of the curve between A and B.

From this definition it is evident that a straight line has no curvature, and that a circle has constant curvature, since the ratio is constant, however long the arc may be. This latter fact may be brought out more clearly by the following investigation.

109. Radius and centre of curvature.

Let AC and BC be the normals to the curve at two adjacent points A and B respectively, meeting at C. Then if we consider the short arc AB as practically identical with the arc of a circle, C will be its centre and CA and CB will be its radius. Denote CA and CB by the letter ρ. C is called the centre of curvature and ρ the radius of curvature at A when B coincides ultimately with A. And the circle whose centre is C and radius ρ is called the circle of curvature at A. It is sometimes called the osculating circle, as it more nearly coincides with the curve in the neighbourhood of the point than any other circle does.



The angle at $C = d\phi$, since the angle between two normals equals the angle between the corresponding tangents.

Now, if $d\phi$ is expressed in circular measure, *i.e.* as a fraction of a radian, we have $d\phi = \frac{ds}{\rho}$, by the definition of circular measure.

Hence
$$\frac{d\phi}{ds} = \frac{1}{\rho},$$

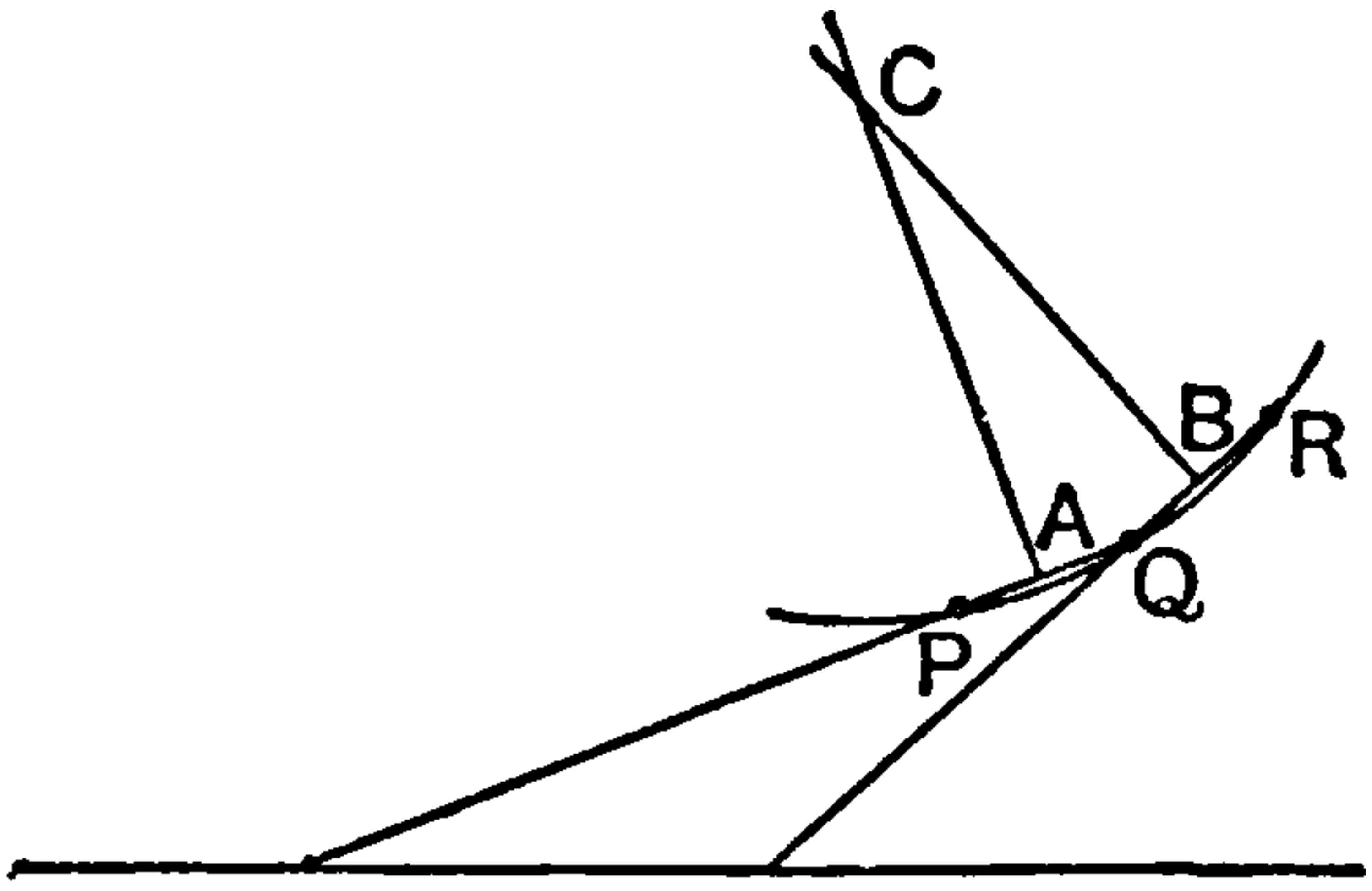
i.e., when the gradient angle is expressed in radians, the curvature is numerically equal to the reciprocal of the radius of curvature.

If the curve *is* a circle, C and ρ are the same for all points, and the curvature of a circle is constant and numerically equal to the reciprocal of its radius. For example, if $\rho = 2$ feet, the curvature $= \frac{1}{2}$, which means that the change of direction is $\frac{1}{2}$ a radian per foot of arc, or 1 radian per 2 feet of arc. If $\rho = 3$ feet, the curvature is 1 radian per 3 feet of arc, or about 19° per foot, since a radian is a little over 57°.

As the equation $\frac{1}{\rho} = \frac{d\phi}{ds}$ is fundamental, we will give a slightly different mode of working it geometrically, which, though not so simple as the foregoing, is in some respects more satisfactory.

Let P, Q, R be three equidistant points on the curve, and let the chords PQ, QR make the angles ϕ and $\phi + d\phi$ respectively with the x axis.

Bisect PQ and QR at right angles by AC and BC. Then C will be the centre of the circle described through P, Q, and R. Let ρ be the radius of the circle. The angle at C $= d\phi$, and if AB is called ds, $d\phi = \frac{ds}{\rho}$.



Hence $\frac{1}{\rho} = \frac{d\phi}{ds}$ where $d\phi$ is the angle between two consecutive normals, and ds is the element of arc intercepted between them.

This is the same as the previous expression for the curvature, since the angle between any two normals is equal to the angle between the corresponding tangents. Moreover, the connection between the above diagrams is very simple, for, without altering the directions of the chords PQ and QR, we can by a slight movement make them the tangents at their middle points A and B respectively.

110. To obtain the radius of curvature as a function of x.

If the equation of the curve is $y = f(x)$, its gradient y_1 will equal $f'(x)$, and the rate of change of gradient per unit change of x will be $\frac{dy_1}{dx}$, which we have denoted by y_2, and which is often written $f''(x)$, to signify that it has been obtained from $f(x)$ by two differentiations. We shall be able to express ρ in terms of y_1 and y_2.

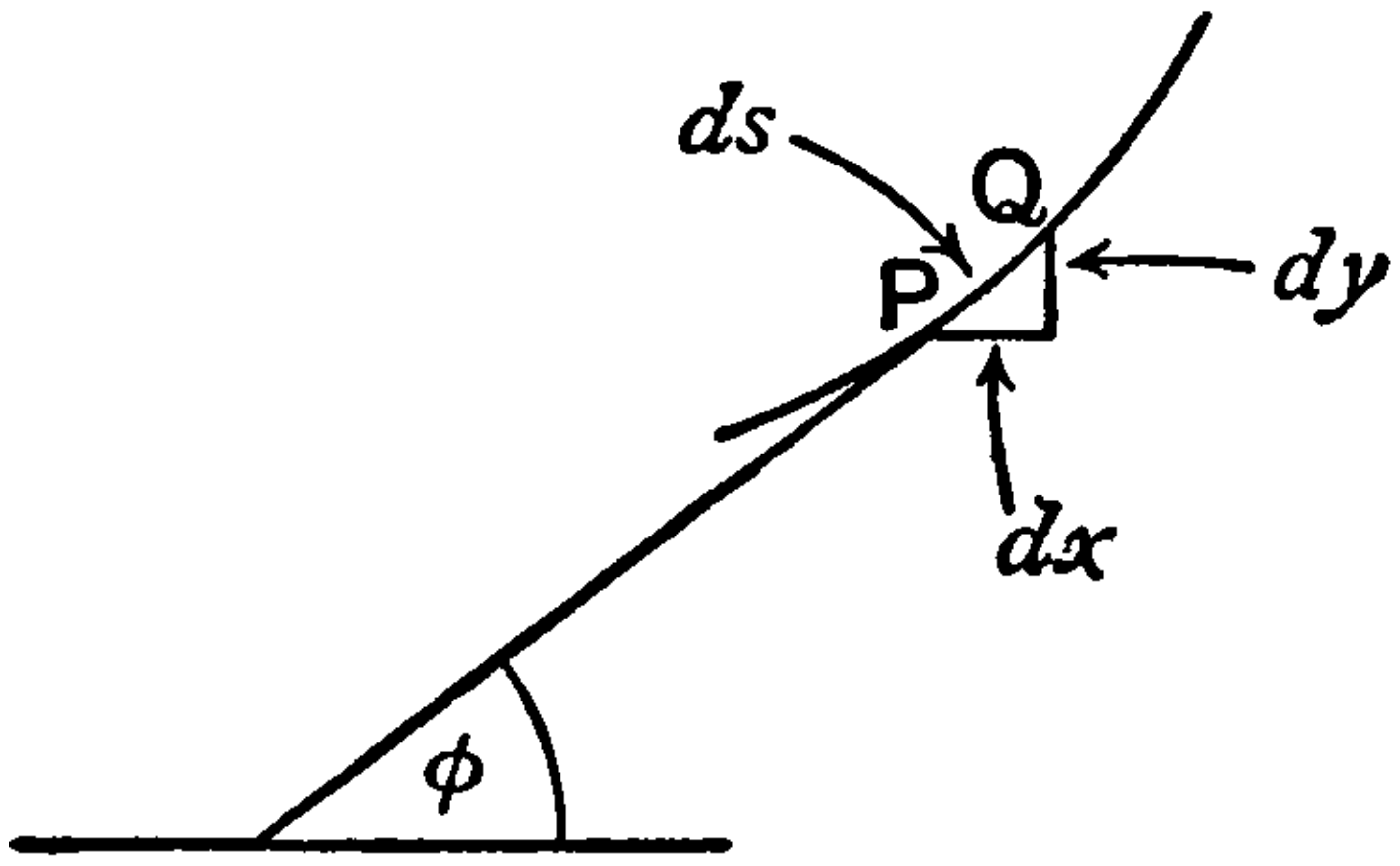
For $$\tan\phi = y_1;$$

$$\therefore \sec^2\phi \frac{d\phi}{dx} = y_2;$$

$$\therefore d\phi = y_2.\cos^2\phi.dx.$$

Also $$\sec\phi = \frac{ds}{dx} \text{ (see figure);}$$

$$\therefore\ ds = \sec\phi . dx.$$



Therefore $$\frac{1}{\rho} = \frac{y_2 \cos^2\phi . dx}{\sec\phi\, dx}$$

$$= y_2 \cos^3\phi$$

and $$\rho = \frac{1}{y_2}\sec^3\phi :$$

i.e. the curvature $= y_2 \cos^3\phi$,

and the radius of curvature is its reciprocal.

To find the radius of curvature at any point we have therefore to find the value of ϕ at the point from the equation $\tan\phi = y_1 = f'(x)$, then we must find the value of y_2 at the point, and multiply the reciprocal of y_2 by $\sec^3\phi$.

The simple connection between y_2 and the curvature is worth considering. The curvature is the rate of change of the gradient-angle ϕ per unit length of arc; and y_2 is the rate of change of the gradient itself, *i.e.* $\tan\phi$, per unit change of abscissa. Tan ϕ increases at a greater rate than ϕ, and the arc increases at a greater rate than the abscissa, hence, for both reasons, y_2 is greater than the curvature. The above investigation shows that we must multiply y_2 by $\cos^3\phi$ to obtain the curvature.

We may put the details of the argument in words: we want to find the rate of change of ϕ per unit length of arc in terms of the rate of change of gradient ($\tan\phi$) per unit change of abscissa. We know that $\frac{d\tan\phi}{dx} = \sec^2\phi\frac{d\phi}{dx}$, that is, the rate of change of gradient per unit change of abscissa is $\sec^2\phi$ times the rate of change of ϕ per unit change of *abscissa.* Also we know that $\frac{ds}{dx}=\sec\phi$, *i.e.* the length of arc between two adjacent points is $\sec\phi$ times the change of abscissa. Therefore, finally, the rate of change of gradient per unit change of abscissa is $\sec^3\phi$ times the rate of change of ϕ per unit length of arc. That is, y_2 is sce$^3\phi$ times the curvature.

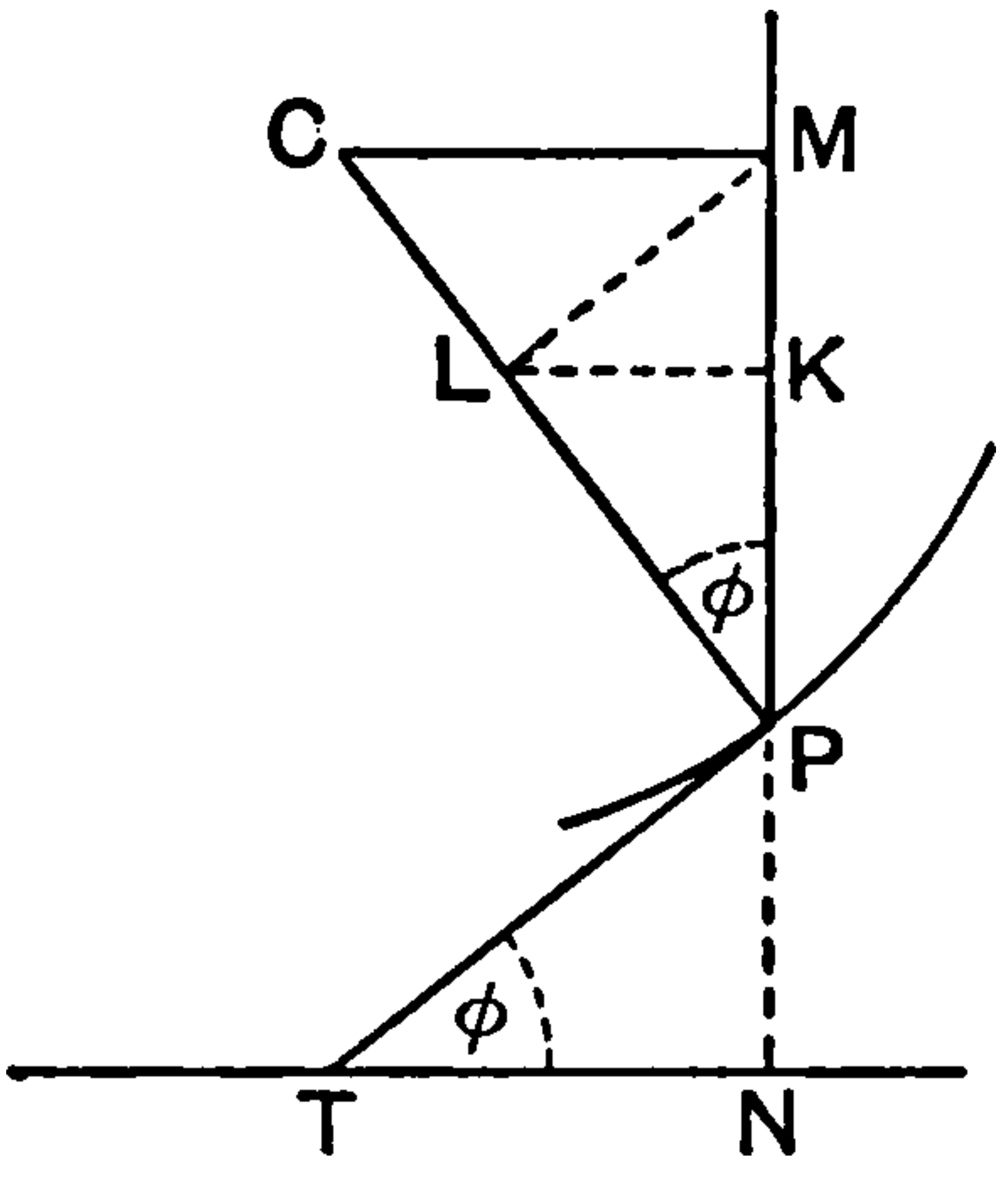
The formula for ρ may be written in the form

$$\rho=\frac{\{1+y_1^2\}^{\frac{3}{2}}}{y_2},$$

since $y_1=\tan\phi$. But it is generally more convenient to use the simpler form $\frac{1}{y_2}\sec^3\phi$.

111. Geometrical construction for ρ.

To find the radius of curvature at any point P of a given curve.



Draw the normal and ordinate at P. On the ordinate take PK, towards the inside of the curve, numerically equal to the reciprocal of y_2. Draw KL perpendicular to PK to cut the normal in L, then $\text{PL}=\text{PK}\sec\phi$; draw LM perpendicular to PL to meet the ordinate in M, then $\text{PM}=\text{PL}\sec\phi=\text{PK}\,\text{sce}^2\phi$; lastly, draw MC perpendicular to the ordinate to cut the normal at C, then $\text{CP}=\text{PK}\sec^3\phi$. Hence, if PK has been drawn to the inside of the curve, C will be the centre of curvature, and CP the radius of curvature, of the given curve at the point P.

112. Direction in which PK must be drawn.

If the curve has been drawn, it will be obvious in which direction to draw PK, since it must be to the inside of the curve. But we can tell, independently of the figure, in which way to draw PK. If y_2 is positive, K will be above P, *i.e.*, PK will be drawn upward, and if y_2 is negative, PK will be drawn downward.

For $$y_2=\frac{dy_1}{dx}=\sec^2\phi\frac{d\phi}{dx},$$

$\therefore$ when y_2 is positive, $\frac{d\phi}{dx}$ is positive, *i.e.*, ϕ will increase with x. This can only be the case if the curve lies above the tangent at P, *i.e.*, if the centre of curvature lies above the tangent. On the other hand, if y_2 is negative, ϕ decreases as x increases, and the curve must lie below the tangent.

The student should convince himself of the general truth of this by taking a part of the curve where the gradient is negative. He must remember that ϕ is the angle made by the upper part of the tangent with the right-hand part of the x-axis, so that ϕ is obtuse when the gradient is negative.

113. Special cases.

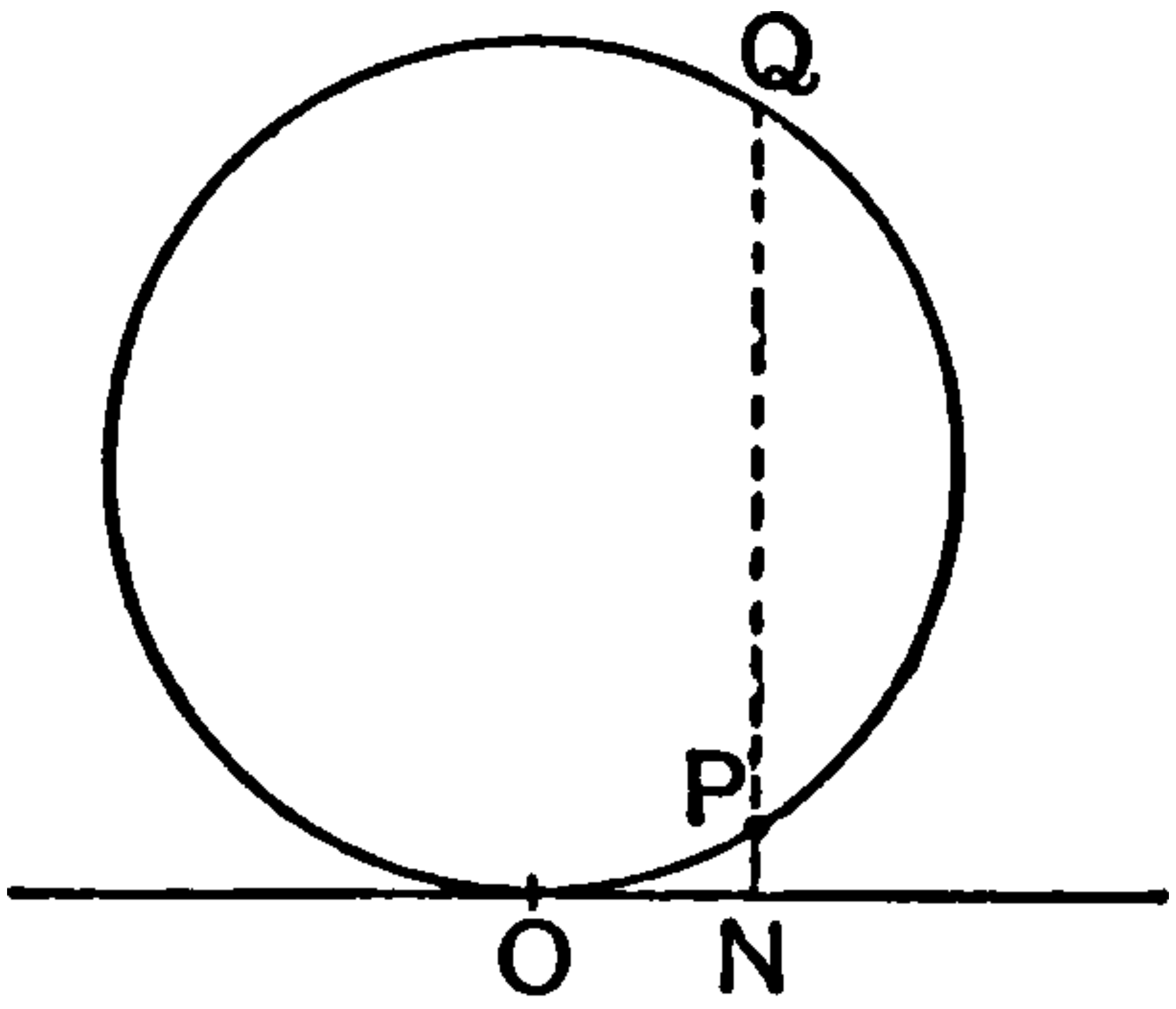
(i.) If $\phi=0$, $\sec^3\phi=1$; $\therefore$ $\rho=\frac{1}{y_2}=\text{PK}$. This is an important case in many practical examples, such, for instance, as the curvature of beams, which is in general so slight that ϕ is sufficiently small at each point for us to take $\sec\phi=1$ without sensible error at each point. It is also true, of course, in all curves at the points where the gradient is zero, as, for example, when y is a maximum or minimum.

(ii.) If y_2 is zero, as at a point of inflexion, PK, and therefore ρ, is infinite, *i.e.*, at a point of inflection a straight line more nearly coincides with the curve than any circle with finite radius can Briefly, at such a point the curve is straight, its curvature being zero.

(iii.) At a point where y_1 is infinite, *i.e.* where the curve is perpendicular to the x-axis, the method fails. To obtain the curvature at such a point, we must invert our treatment and treat x as a function of y, obtaining $x_1=\frac{dx}{dy}$, and $x_2=\frac{dx_1}{dy}$. Of course x_1 will be zero at such a point, and the curvature will then be x_2, and $\rho=\frac{1}{x_2}$, the radius being drawn to the right if x_2 is positive, and to the left if x_2 is negative. We think, in this case, of the gradient angle as being made by the tangent with the y-axis, *i.e.* by the right-hand part of the tangent with the upper part of the y-axis.

114. Value of ρ at the origin when the x-axis is the tangent.

We may avoid the calculus in this case by the following method.



The curvature of a curve at any point is the same as the curvature of its circle of curvature at that point; hence we need only consider the circle of curvature, letting it take the place of the curve.

Let the figure show the circle of curvature at O, and let P be an adjacent point on the circle with coordinates x, y, *i.e.* ON $=x$, and NP $=y$.

Then, if NP cuts the circle again in Q, we have NP.NQ $=$ ON^2; but NQ is ultimately equal to 2ρ;

$$\therefore\ y.2\rho=x^2,$$

i.e.
$$\rho=\frac{x^2}{2y}$$

in the limit when x and y are infinitely small.

Thus, in the parabola $y=x^2$,

$$\rho=\frac{x^2}{2y}=\tfrac{1}{2}.$$

Similarly, if the curve touches the y-axis at the origin,

$$\rho=\frac{y^2}{2x}.$$

115. Mode of finding ρ when the vertical and horizontal scales are different.

In the above investigation, and particularly in the construction of Art. 111, we have supposed the figure drawn strictly to one scale, *i.e.* the same vertical and horizontal scales. But in many cases it has been found convenient to take the vertical scale different from the horizontal. In that case several difficulties arise. First, the circle of curvature, if plotted by points, must also be plotted with its vertical scale different from the horizontal, which would distort it into an ellipse; or if we plot $\frac{1}{y_2}$ and try to find the centre of curvature geometrically, the difficulty arises as to what scale we must plot it on. Of course there *must* be a true circle of curvature at each point—the problem is how to modify our construction so as to find its centre.

If we take the numerical equation $y=f(x)$, (to fix the ideas the student might have in his mind the parabola $y=x^2$ plotted on p. 6 with vertical scale 1-tenth of the horizontal), and consider how we plot the various points for calculated numerical values of x and y, we see that we take an arbitrary horizontal unit of scale and plot the abscissae as multiples of that unit, and then we plot the values of y as multiples of an arbitrary vertical unit. Let us take a to denote the length of the horizontal unit, and b to denote the length of the vertical unit, then if (*e.g.*) we plot the point $x=2$, $y=4$, we really take $x=2a$ horizontally, and $y=4b$ vertically, so that $\frac{x}{a}=2$, and $\frac{y}{b}=4$, when x and y are the plotted lengths. And so in the case of every other point. The equation gives the connection between $\frac{x}{a}$ and $\frac{y}{b}$. Consequently the full expression for the curve plotted is $\frac{y}{b}=f\left(\frac{x}{a}\right)$, where x and y denote the actual lengths plotted. [In the case of the parabola the equation from which we plotted was $\frac{y}{b}=\left(\frac{x}{a}\right)^2$ where $a=10b$.]

If then we find the radius of curvature at each part of the curve

$$\frac{y}{b}=f\left(\frac{x}{a}\right),$$

and then make use of the relation between a and b, we shall have the value of ρ required.

Differentiating, we have

$$\frac{y_1}{b}=\frac{1}{a}f'\left(\frac{x}{a}\right);$$

$$\therefore\ y_1=\frac{b}{a}f'\left(\frac{x}{a}\right)=\tan\phi.$$

Again, $$y_2=\frac{b}{a^2}f''\left(\frac{x}{a}\right);$$

$$\therefore\ \rho=\frac{a^2}{bf''\left(\frac{x}{a}\right)}\sec^3\phi.$$

To construct for ρ, we must draw the vertical line $\frac{a^2}{bf''\left(\frac{x}{a}\right)}$, which call k, and then by the zigzag construction find the centre of curvature.

If we plot k on the vertical scale, its numerical value is $\frac{k}{b}$, which, from the above equation, $=\frac{a^2}{b^2f''\left(\frac{x}{a}\right)}$.

Applying this to the curve $y=x^2=f(x)$, we find

$$f'(x)=2x \quad \text{and} \quad f''(x)=2;$$

$$\therefore\ \frac{k}{b}=\frac{a^2}{2b^2}.$$

Now in the diagram, p. 6,

$$a=10b;$$

$$\therefore\ a^2:b^2=100;$$

$$\therefore\ \frac{k}{b}=50,\text{ or } k=50b.$$

Hence, to find ρ for any point P on the curve, we must draw k vertically upwards to a length of 50 vertical units from P, then draw the normal, and find C by the zigzag construction.

In particular, the centre of curvature at the origin is 50 units up on the axis of y. This can be easily tested by the student by actual trial on the diagram with a pair of compasses.

NOTE.—To find the radius of curvature of the parabola at the origin we might have used Art. 114. Since the axis of x is the tangent there, $\rho = \frac{x^2}{2y}$; and since $\frac{y}{b} = \frac{x^2}{a^2}$, it follows at once that $\rho = \frac{a^2}{2b}$. Hence its numerical value on the vertical scale is $\frac{a^2}{2b^2}$.

116. The reasoning by which the formula $k = \frac{a^2}{bf''(x)}$ was established is a little troublesome at first, as all reasoning in connection with change of units must be; but if we remember that $y = f(x)$ is merely a numerical equation between x and y, and that what we plot are lengths corresponding to those numbers, we see that the whole difficulty consists in ascertaining what is the real connection between the lengths and the numbers represented by them. The fundamental relation is as follows:—If a length x is represented by a number when x is measured in terms of a particular unit, the number expresses the ratio of the length of x to the length of the unit. Thus, if the number is 2 when the length of the unit is a, $2 = x : a$. *E.g.*, let $x = 24$ inches; then if x is measured in inches, the number representing it is 24, whereas if x is measured in feet, the number representing it is 2, *i.e.* 24 is the ratio of x to 1 inch, while 2 is the ratio of x to 1 foot. Similarly, if y is measured in terms of a unit whose length is b, the numerical measure of y is the ratio of y to b.

Consequently, if $y = f(x)$ is a *numerical* equation, the curve which we actually plot to represent this equation is $\frac{y}{b} = f\left(\frac{x}{a}\right)$ when a is the length of the horizontal unit of scale and b is the vertical unit.

No difficulty of this sort ever arises if $b=a$, but in many cases such equality is practically inconvenient.

117. We will, as another example, find the radius of curvature of the curve $y=x(5-x)$, (1) at the highest point, (2) at the points where the curve cuts the x-axis. We will first consider the two scales equal, and then take $a=5b$ where a, b are the units of the horizontal and vertical scales.

$$y=x(5-x)=5x-x^2,$$
$$y_1=5-2x,$$
$$y_2=-2;$$

$\therefore$ at the highest point, where $\phi=0$, $\rho=\frac{1}{2}$, if the scales are equal.

The negative value of y_2 shows that the centre of curvature is below the curve. There is no object in attaching a sign to the value of ρ.

If $a=5b$, we must take the formula $\frac{\rho}{b}=\frac{a^2}{b^2y_2}=-\frac{25}{2}$;

$$\therefore\ \rho=12\tfrac{1}{2} \text{ vertical units.}$$

At the points where the curve cuts the axis of x, $x(5-x)=0$;

$$\therefore\ x=0, \text{ or } 5;$$

$$\therefore\ \tan\phi=\pm 5, \text{ and } \sec\phi=\pm\sqrt{26}, \text{ if the scales are equal;}$$

$$\therefore\ \rho=\frac{26\sqrt{26}}{2}=13\sqrt{26} \text{ units.}$$

If the scales are unequal, $\pm\tan\phi=5$ vertical units to 1 horizontal unit

$$=\frac{5b}{a}=1, \text{ if } a=5b;$$

whence $\pm\sec\phi=\sqrt{2}$

and $\frac{k}{b}=\frac{a^2}{b^2y_2}=-\frac{25}{2}$;

$$\therefore\ \rho=\frac{25}{2}\times 2\sqrt{2}b=25\sqrt{2}b=5\sqrt{2}a.$$

The student will find it helpful to notice, if he is drawing this curve with the scale relation $a=5b$, that the centre of curvature of the origin has the abscissa $5a$, and the ordinate $-5a$, or $-25b$, and the centre of curvature of the point $(5a, 0)$ is at $(0, -25b)$. If he draws arcs of these two circles, and also the arc of the circle of curvature passing through the highest point, (found above), he will find it very easy to draw in the portion of the curve lying above the x-axis.

For a graph of this curve see answer to Ex. 1, p. 15.

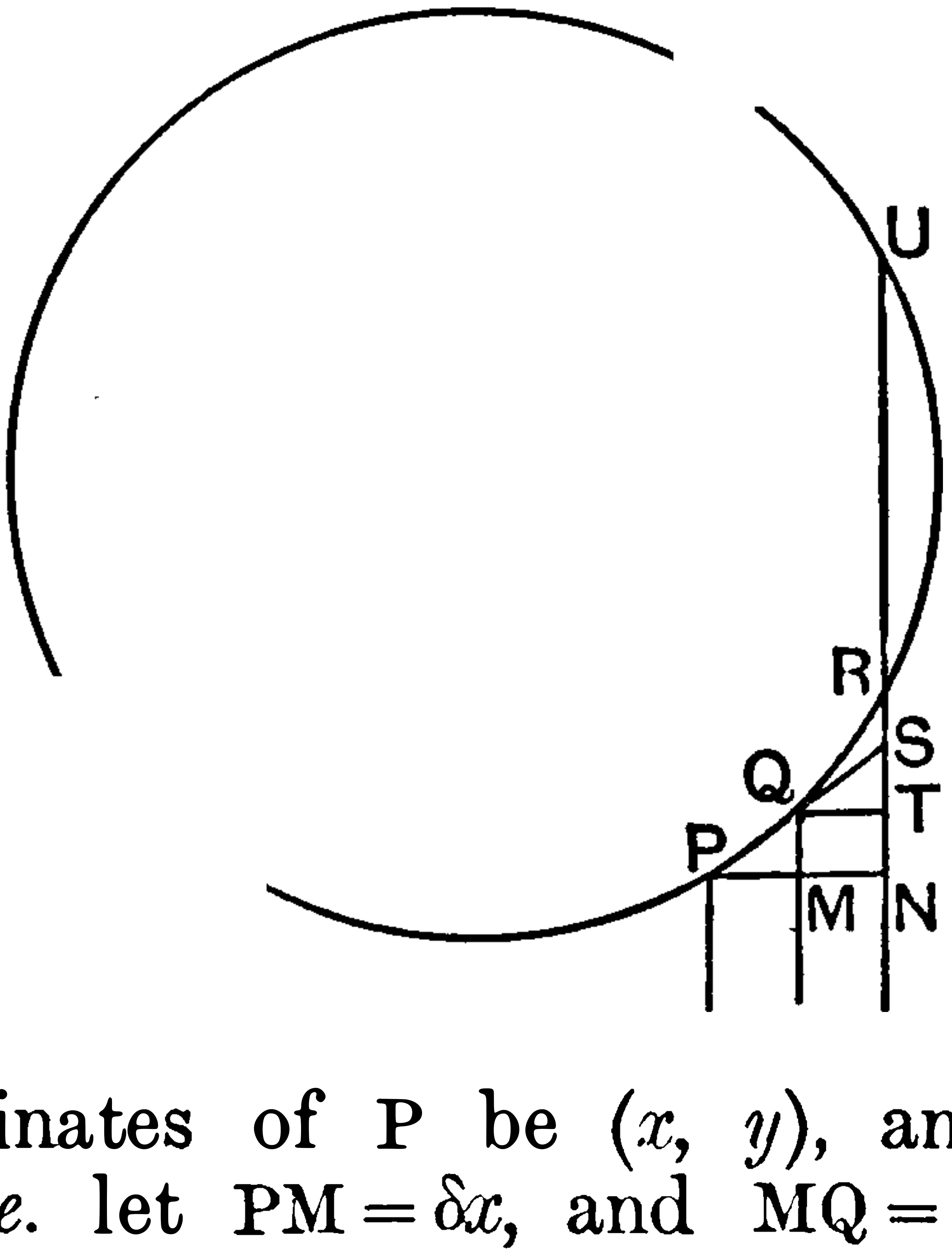
118. Chord of curvature.

If we draw the circle of curvature corresponding to any point P of a curve, the chords of this circle drawn through P are called chords of curvature at P. The chord drawn along the normal is the diameter of curvature; other important chords are the vertical and horizontal chords of curvature.

The vertical chord $= 2\rho \cos \phi = 2k \sec^2\phi$, where $k = \dfrac{1}{y_2}$; and the horizontal chord $= 2\rho \sin \phi = 2k \sec^2\phi \tan \phi$.

The following geometrical investigation of the length of the vertical chord is instructive, and will incidentally throw a good deal of light on the meaning of y_2.

Let P, Q, R be three neighbouring points on the given curve such that the ordinate of Q is midway between those of P and R, and let a circle be drawn through P, Q, R. This will ultimately be the circle of curvature at P when Q and R coincide with P.



Let the coordinates of P be (x, y), and those of Q be $(x + \delta x, y + \delta y)$, *i.e.* let $PM = \delta x$, and $MQ = \delta y$. The abscissa of R is $x + 2\delta x$. The ordinate of R exceeds the ordinate of Q by the amount TR, of which $TS = \delta y$. Now $QM = \delta y$ where y is the ordinate of P. ∴ the corresponding notation for RT is $\delta(y + \delta y)$ where $y + \delta y$ is the ordinate of Q. We may separate this expression into the two parts $\delta y + \delta(\delta y)$, of which $TS = \delta y$ and $SR = \delta(\delta y)$.

The notation $\delta(\delta y)$ expresses therefore the excess of the increase of ordinate as we pass from Q to R over the increase of ordinate as we pass from P to Q. It is often written, briefly, $\delta^2 y$ (*read* delta-squared y).

Hence $\delta^2 y$ denotes the vertical height of R above the chord PQ.

Now produce the ordinate of R to meet the circle in U. Then RU will ultimately be the vertical chord of curvature at P, and so also will SU; since both R and S ultimately coincide with P. Let PQ be denoted by δs, since it is ultimately equal to the small arc PQ. Then $SQ = \delta s$ and $SP = 2\delta s$.

Now
$$SR.SU = SP.SQ;$$
$$\therefore\ \delta^2 y.SU = 2(\delta s)^2$$
and
$$\frac{\delta^2 y}{(\delta x)^2}.SU = 2\left(\frac{\delta s}{\delta x}\right)^2.$$

Now let Q and R move into coincidence with P, whereupon $\frac{\delta s}{\delta x}$ becomes $= \sec\phi$ if ϕ is the gradient angle at P of the curve and circle, and at the same time $\frac{\delta^2 y}{(\delta x)^2}$ is usually written $\frac{d^2 y}{(dx)^2}$ or $\frac{d^2 y}{dx^2}$ (*read* d-squared y by dx-squared).

Hence
$$\frac{d^2 y}{dx^2}.SU = 2\sec^2\phi,$$
where SU is now the vertical chord of curvature at P.

Now, evidently, from the figure, $\frac{d^2 y}{dx^2} = \frac{d\tan\phi}{dx}$, for $d\tan\phi$ means the excess of the gradient of QR above that of PQ, which is $d\left(\frac{dy}{dx}\right)$, *i.e.* $\frac{d(dy)}{dx}$, since dx is the same between Q and R as it was between P and Q, that is, dx is constant. Hence $d(dy)$ means exactly the limiting value of $\delta(\delta y)$ or RS, and dx means the limiting value of δx or PM. Therefore $\frac{d\tan\phi}{dx}$ is the limiting value of $\frac{\delta(\delta y)}{(\delta x)^2}$.

Hence $\frac{d^2 y}{dx^2}$ is what we have before denoted by y_2, and we have the formula,

$$\text{vertical chord of curvature} = \frac{2\sec^2\phi}{y_2} = 2k\sec^2\phi, \text{ as before.}$$

EXAMPLES.

1. Find the radius of curvature at the point $(2, \frac{1}{2})$ of the curve $y=\frac{1}{x}$,

(1) when the vertical and horizontal scales are equal;

(2) when the horizontal scale $=\frac{1}{2}$ the vertical.

2. Find the radius of curvature of the above curve at the point $(1, 1)$ under the same scale conditions.

3. Find the radius of curvature at the point $(1, 2)$ of the curve $y=x+\frac{1}{x}$ when the scales are equal.

4. Find the radius of curvature at any point of the curve $y=e^x$, and also at any point of the curve $y=be^{\frac{x}{a}}$.

5. Find the minimum radius of curvature in the curves of No. 4.

6. Find the radius of curvature of the curve $y=a\cosh\frac{x}{a}$ at any point, and show that it is equal to the length of the portion of the normal at the point intercepted between the curve and the axis of x.

7. Find the radii of curvature at the points of zero gradient on the curve $y=x(10-x)^2$.

8. Find the radii of curvature at the points of zero gradient on the curve $y=x(144-x^2)$.

9. Find the radius of curvature of the curve $y=x^3-4x^2-11x+32$ at the point $(2, 2)$, (1) when the vertical and horizontal scales are equal; (2) when the horizontal scale is 5 times the vertical scale.

10. Find the radii of curvature of the curve $y=x^2(6-x)$, (1) at the origin, (2) at the highest point between $x=0$ and $x=6$; when the horizontal scale is 5 times the vertical.

11. In any curve $y=f(x)$, prove that $y_2 \div x_2 = -y_1^3$.
Exemplify in the case of $y=x^2$.

12. Find the radius of curvature at any point of the ellipse

$$x=a\cos\theta,\quad y=b\sin\theta.$$

13. Find the radius of curvature at any point of the hyperbola

$$x=a\sec\theta,\quad y=b\tan\theta.$$

14. Show that the equation of the circle of curvature of the parabola $y^2=4ax$, at the origin, is $x^2+y^2=4ax$.

15. Show that the equation of the circle of curvature of the conic $\mathrm{A}x^2+\mathrm{B}y^2=2\mathrm{C}x$, at the origin, is $\mathrm{B}x^2+\mathrm{B}y^2=2\mathrm{C}x$.

16. Deduce the length of the radius of curvature, at $(-a, 0)$, of the conics $b^2x^2 \pm a^2y^2=a^2b^2$, by moving the origin to this point.

17. Prove that the line drawn from (a, b) perpendicular to the line $bx+ay=ab$ passes through two of the principal centres of curvature of the ellipse $b^2x^2+a^2y^2=a^2b^2$.

[The principal centres of curvature are those which correspond to the ends of the major and minor axes of the ellipse.]

CHAPTER IX.

SUCCESSIVE DIFFERENTIATION—INTEGRATION.

119. Successive Differential Coefficients of a Function. If y is a function of x, so also (in general) is its differential coefficient, and consequently this also has a differential coefficient which is also a function of x, and so on. (The only exception occurs when one of the differential coefficients is a constant, and the rest, consequently, become zero, and none of these are functions of x.)

These successive differential coefficients are called the first, second, third, etc. Thus, if the function is x^5, the first differential coefficient is $5x^4$, the second is $5.4x^3$, the third is $5.4.3x^2$, etc.

There are several systems of notation which may be employed to denote these successive functions. The most usual are:

$$y, \quad \frac{dy}{dx}, \quad \frac{d^2y}{dx^2}, \quad \frac{d^3y}{dx^3}, \ldots,$$

$$y, \quad \mathrm{D}y, \quad \mathrm{D}^2y, \quad \mathrm{D}^3y, \quad \ldots,$$

$$f(x), \quad f'(x), \quad f''(x), \quad f'''(x), \quad \ldots,$$

but sometimes, for temporary convenience, a brief, incomplete, notation may be used, such as

$$y, \quad y_1, \quad y_2, \quad y_3, \quad \ldots,$$

or

$$y, \quad y', \quad y'', \quad y''', \quad \ldots.$$

As an example of successive differential coefficients we may take

$$\begin{aligned} y &= x^5, \\ \mathrm{D}y &= 5x^4, \\ \mathrm{D}^2y &= 5.4x^3 = 20x^3, \\ \mathrm{D}^3y &= 5.4.3x^2 = 60x^2, \\ \mathrm{D}^4y &= 5.4.3.2x = 120x, \\ \mathrm{D}^5y &= \lfloor\!\underline{5} = 120, \\ \text{and} \qquad \mathrm{D}^6y &= 0, \text{ etc.} \end{aligned}$$

In this case, therefore, the first four differential coefficients are functions of x, the fifth is a constant, and all the succeeding ones are zero. If, however, the original function had been anything except a positive integral power of x, *all* its differential coefficients would have been also functions of x.

EXAMPLES.

1. If $y = 15x^4 - 7x^2 + 3$, find $\frac{d^4y}{dx^4}$.

2. If $f(x) = x^{\frac{5}{2}} - 7x^{\frac{3}{2}} + 8x^{\frac{1}{2}}$, find $f^{\mathrm{iv}}(x)$.

3. If $y = \sin x$, show that $\mathrm{D}^n y = \sin\left(x + n\frac{\pi}{2}\right)$.

4. If $y = \sin 3x$, show that $\mathrm{D}^5 y = 3^5 \sin\left(3x + \frac{5\pi}{2}\right)$.

5. If $y = 5x^5 + 3x^4 + 2x^3 + x^2 - x + 6$, show that $\mathrm{D}^6 y = 0$.

6. If $y = ax^{s-1} + bx^{s-2} + \ldots$, show that $\mathrm{D}^s y = 0$, if s is a positive integer.

120. Successive Integrals of a Function. It is important to notice that such a series as

$$x^5,\ 5x^4,\ \ldots$$

can be not only continued forwards, as above, by successive differentiation, but can also be continued backwards, the terms on the left of x^5 being such that x^5 is one of their differential coefficients. Such an extended series is

$$\ldots,\ \frac{x^8}{8.7.6},\ \frac{x^7}{7.6},\ \frac{x^6}{6},\ \mathbf{x^5},\ 5x^4,\ \ldots.$$

The terms on the left of x^5 are termed its *integrals*, its first integral being $\frac{x^6}{6}$, its second being $\frac{x^7}{7.6}$, and so on.

There are several notations for the successive integrals of a function y, but the only one that can usefully be given at this stage is

$$\ldots,\ \mathrm{D}^{-3}y,\ \mathrm{D}^{-2}y,\ \mathrm{D}^{-1}y,\ y.$$

Thus, if

$$y = x^5,$$

$$\mathrm{D}^{-1}y = \frac{x^6}{6},$$

$$\mathrm{D}^{-2}y = \frac{x^7}{7.6} = \frac{x^7}{42},$$

etc.

These integral functions are, however, not so definite as the differential coefficients are, for not only is $\frac{1}{6}x^6$ an integral of x^5, but so also is $\frac{1}{6}x^6 + a$, where a is any constant. For, if we take $\mathrm{D}(\frac{1}{6}x^6 + a)$, the result is x^5, since $\mathrm{D}a = 0$;

$$\therefore\ \mathrm{D}^{-1}(x^5) = \tfrac{1}{6}x^6 + a.$$

Similarly,

$$\mathrm{D}^{-2}(x^5) = \tfrac{1}{42}x^7 + ax + b,$$

and

$$\mathrm{D}^{-3}(x^5) = \tfrac{1}{336}x^8 + ax^2 + bx + c,$$

(where a, b, c are in each case *any* constants, not necessarily the same in each case. They are called arbitrary constants.)

These results can be verified at once by taking respectively the second and the third differential coefficients of these two functions

It will be seen that the first integral of a function may contain *one* arbitrary constant, the second may contain *two*, and so on. These integrals are called the *general* integrals of x^5, and the particular cases in which special values are given to a, b, c, ... are called *particular* integrals, and in especial the integrals in which a, b, c, ... are all zero, may be called the *principal* integrals. Thus $\frac{1}{336}x^8$ is the principal third integral of x^5. The remaining part, viz , $ax^2 + bx + c$, may be called the *complementary* part of the third integral. It may be written as $\mathrm{D}^{-3}(0)$, for if it be differentiated 3 times, the result $= 0$;

$$\therefore\ \mathrm{D}^{-3}(x^5) = \tfrac{1}{336}x^8 + \mathrm{D}^{-3}(0).$$

121. *To find a general formula for* $D^r(x^n)$. The results indicated in the above example may be easily extended into a general formula for $D^r(x^n)$, where n is a positive integer, and r is either positive or negative.

For, if

$$\begin{aligned} y &= x^n; \\ Dy &= nx^{n-1}; \\ D^2y &= n(n-1)x^{n-2}; \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ D^ry &= n(n-1)(n-2)\ldots(n-r+1)x^{n-r}, \\ &= \frac{\underline{|n}}{\underline{|n-r}}x^{n-r}. \end{aligned}$$

This is worked on the supposition that r is a positive integer, but it holds equally well if r is a negative integer, as will be seen at once by taking particular cases.

Thus, if $r = -1$,

$$D^{-1}y = \frac{\underline{|n}}{\underline{|n+1}}x^{n+1} = \frac{x^{n+1}}{n+1}.$$

This only means that $D\left(\dfrac{x^{n+1}}{n+1}\right) = y$.

Again,

$$D^{-2}y = \frac{\underline{|n}}{\underline{|n+2}}x^{n+2} = \frac{x^{n+2}}{(n+2)(n+1)},$$

which means that $y = D^2\dfrac{x^{n+2}}{(n+2)(n+1)}$, the truth of which is evident on actually differentiating twice.

Of course these are only the principal integrals. The general formula is

$$D^ry = \frac{\underline{|n}}{\underline{|n-r}}x^{n-r} + D^r(0)$$

where $D^r(0) = 0$ if r is a positive integer, but if r is negative, $= -s$ suppose,

$$D^r(0) = D^{-s}(0) = ax^{s-1} + bx^{s-2} + \ldots + \text{ a constant},$$

as can be easily verified by seeing that $D^s(ax^{s-1} + bx^{s-2}\ldots) = 0$ (see Ex. 6, p. 125).

NOTE.—When r is a positive integer, the formula presents no difficulty if r is less than n, but when r is equal to or greater than n, the denominator $\underline{|n-r}$ requires special interpretation, as when $r=n$, $n+1$, $n+2\ldots$ this factorial becomes $\underline{|0}$, $\underline{|-1}$, $\underline{|-2}, \ldots$. Now we know that $\mathrm{D}^n x^n = \underline{|n}$, hence on comparing this known result with that given by the formula, we see that we must put $\underline{|0} = 1$. All the differential coefficients beyond the n^{th} are zero, hence we must put $\underline{|-1} = \infty$, $\underline{|-2} = \infty$, if the formula is to give the correct result. Of course we cannot attach a direct meaning to any of these quantities $\underline{|0}$, $\underline{|-1}, \ldots$. All we can do is to treat them as algebraic extensions of the factorial series, and make them obey the formal laws of factorials, which are (1) that $\underline{|1} = 1$, and (2) that $\underline{|n} = n\underline{|n-1}$.

This will give us $\underline{|1} = 1.\underline{|0}$, $\quad \therefore \underline{|0} = 1$;

and $\underline{|0} = 0\underline{|-1}$, $\quad \therefore \underline{|-1} = \infty$.

Also $\underline{|-1} = -1\underline{|-2}$, $\quad \therefore \underline{|-2} = \infty$,

and so on, which exactly agree with the values required above.

Armed with these results we may affirm generally that

$$\mathrm{D}^r(x^n) = \frac{\underline{|n}}{\underline{|n-r}} x^{n-r} + \mathrm{D}^r(0)$$

for all integer values of r, whether positive or negative, provided n is a positive integer.

It is possible so to define $\underline{|n}$ that the same formula will hold when n is a negative integer, or even when n is fractional, but it is easy to establish a more convenient formula for the former case, and the latter is too difficult to be attempted here. The student should, however, realize that generalizations of formulae to cover all possible cases are of great mathematical interest, and are continually suggesting themselves for contemplation. Indeed, mathematicians have investigated the case when r is fractional, though we can certainly get no simple idea of what is meant by fractional differentiation.

122. *To find a general formula for* $D^r(x^{-n})$, *where* n *is a positive integer.*

Let $$y = x^{-n};$$

$$\therefore\ Dy = -nx^{-n-1},$$

$$D^2y = n(n+1)x^{-n-2},$$

$$D^ry = (-1)^r n(n+1)\ldots(n+r-1)x^{-n-r};$$

i.e. $$D^r\left(\frac{1}{x^n}\right) = \frac{\underline{|n+r-1}}{\underline{|n-1}}\cdot\frac{(-1)^r}{x^{n+r}}.$$

This formula also holds good for integrations. For it gives

$$D^{-1}\left(\frac{1}{x^n}\right) = -\frac{\underline{|n-2}}{\underline{|n-1}}\cdot\frac{1}{x^{n-1}} = -\frac{1}{(n-1)x^{n-1}},$$

which is evidently a function whose differential coefficient is x^{-n}.

The complete integral is

$$-\frac{1}{(n-1)x^{n-1}} + D^{-1}(0),$$

where $D^{-1}(0)$ denotes a constant.

Similarly

$$D^{-s}\left(\frac{1}{x^n}\right) = \frac{\underline{|n-s-1}}{\underline{|n-1}}\cdot\frac{(-1)^s}{x^{n-s}} + D^{-s}(0).$$

This formula holds up to $s = n-1$ inclusive, but fails to take us back any further, that is, it holds until we reach a multiple of x^{-1}; but it will not give us the integral of this function, which (as we know from Art. 63) is $\log x$, and cannot be included in the above series.

In fact, the series of integrals and differential coefficients of x^n and of x^{-n} when n is a positive integer are two distinct series with an impassable gap between them. Thus, taking $n = 2$, we have

$$\ldots,\ \frac{x^5}{60},\ \frac{x^4}{12},\ \frac{x^3}{3},\ |x^2|,\ 2x,\ 2,\ 0,\ \ldots$$

and

$$\ldots,\ -\log x,\ -\frac{1}{x},\ \left|\frac{1}{x^2}\right|,\ -\frac{2}{x^3},\ \frac{6}{x^4},\ -\frac{24}{x^5},\ \ldots.$$

The integrals of $\log x$ will never lead us to the first series, neither will the differential coefficients of the first series lead to the second series.

If, however, n is fractional, there is *one* continuous series from one end of the infinite range to the other, and there is no discontinuity.

Thus, $$\mathrm{D}x^{\frac{1}{2}} = \tfrac{1}{2}x^{-\frac{1}{2}},$$

and the integrals of $x^{\frac{1}{2}}$ give an infinite series of positive powers, while its differential coefficients give an infinite series of negative powers.

123. *To find* $\mathrm{D}^r \log x$, *when* r *is a positive integer.*

If $$y = \log x,$$
$$\mathrm{D}y = x^{-1},$$
$$\mathrm{D}^2 y = -1.x^{-2},$$
$$\mathrm{D}^3 y = -1.-2.x^{-3},$$
$$\dots\dots\dots\dots\dots\dots$$

and $$\mathrm{D}^r y = (-1)^{r-1}\underline{|r-1}\,x^{-r};$$

i.e. $$\mathrm{D}^r \log x = \frac{(-1)^{r-1}\underline{|r-1}}{x^r}.$$

The integrals are not obtainable by this formula. One method of finding them will be given in the following articles. For the usual methods, a book on the integral calculus should be consulted.

124. *To find the integral of* $\log x$.

We can form the series of integrals of $\log x$ by making use of the identity

$$\log_e x = \frac{x^h - 1}{h} \text{ when } h = 0. \quad \text{(Art. 62.)}$$

Thus, to find $\mathrm{D}^{-1} \log x$.

If $$y = \frac{x^h - 1}{h},$$

$$\begin{aligned}
\text{D}^{-1}y &= \frac{1}{h}\text{D}^{-1}(x^h - 1) \\
&= \frac{1}{h}\left(\frac{x^{h+1}}{h+1} - x\right) \\
&= \frac{x}{h+1}\left(\frac{x^h - 1}{h} - 1\right) \\
&= x(\log x - 1) \text{ in the limit when } h = 0.
\end{aligned}$$

This is the first integral of $\log_e x$.

The second integral will be $\text{D}^{-1}(x \log x) - \text{D}^{-1}x$.

The most useful way of finding this will be to establish a formula for $\text{D}^{-1}(x^n \log x)$, and then apply it to the case when $n = 1$, as we shall want the formula for several values of n before we succeed in obtaining the whole series.

125. *To find the integral of $x^n \log x$, and the successive integrals of $\log x$.*

Writing $\frac{x^h - 1}{h}$ for $\log x$, we have

$$\begin{aligned}
\text{D}^{-1}\frac{x^{n+h} - x^n}{h} &= \frac{1}{h}\left(\frac{x^{n+h+1}}{n+h+1} - \frac{x^{n+1}}{n+1}\right) \\
&= \frac{x^{n+1}}{n+h+1}\left(\frac{x^h - 1}{h} - \frac{1}{n+1}\right);
\end{aligned}$$

$\therefore$, putting $h = 0$,

$$\text{D}^{-1}(x^n \log x) = \frac{x^{n+1}}{n+1}\left(\log x - \frac{1}{n+1}\right).$$

We can now find the whole series of integrals of $\log x$. We have proved that

$$\text{D}^{-1}\log x = x \log x - x;$$

$$\begin{aligned}
\therefore\ \text{D}^{-2}\log x &= \text{D}^{-1}(x \log x) - \text{D}^{-1}x \\
&= \frac{x^2}{2}\left(\log x - \frac{1}{2}\right) - \frac{x^2}{2} \\
&= \frac{x^2}{2}\log x - \frac{x^2}{2}\left(1 + \frac{1}{2}\right);
\end{aligned}$$

$$\therefore\ \mathrm{D}^{-3}\log x=\frac{x^3}{2.3}\left(\log x-\frac{1}{3}\right)-\frac{x^3}{2.3}\left(1+\frac{1}{2}\right)$$

$$=\frac{x^3}{1.2.3}\log x-\frac{x^3}{1.2.3}\left(1+\frac{1}{2}+\frac{1}{3}\right).$$

Proceeding in this way, we find

$$\mathrm{D}^{-r}\log x=\frac{x^r}{\underline{|r}}\left(\log x-1-\frac{1}{2}-\frac{1}{3}-\ldots-\frac{1}{r}\right).$$

EXAMPLES.

1. Write down four of the successive differential coefficients, and four of the successive integrals, of

(1) x^4-3x^2+2x-1.

(2) $\frac{1}{x^5}-\frac{2}{x^3}+\frac{1}{x}$.

(3) $\frac{1}{(x+a)^5}$.

(4) $x^{\frac{1}{2}}$.

2. Find a formula for the n^{th} differential coefficient of $\frac{1}{x-a}$, and also for the n^{th} differential coefficient of $\frac{1}{a-x}$.

3. Find a formula for the n^{th} differential coefficient, and for the n^{th} integral, of (1) $\sqrt{x}$; (2) $\frac{1}{\sqrt{x}}$.

4. From the identity $\underline{|n}=n\underline{|n-1}$, prove that

$$\underline{|-m}\div\underline{|-1}=\frac{(-1)^{m-1}}{\underline{|m-1}},$$

when m is a positive integer.

(Note that the factorials of negative integers are infinite, but this formula shows that the ratio between two of them is finite.)

5. Deduce the formula for $\mathrm{D}^r(x^{-m})$ from the formula

$$\mathrm{D}^r(x^n)=\frac{\underline{|n}}{\underline{|n-r}}x^{n-r}.$$

6. Can you assign a meaning or a value to $\underline{|n}$ when n is fractional? Assuming it to have a meaning, deduce from the general formula for $\mathrm{D}^r(x^n)$ the fourth differential coefficient of $\sqrt{x}$.

7. Find the r^{th} differential coefficient, and the r^{th} integral, of

$$(1)\ \frac{x^n}{\underline{|n}};\ (2)\ \frac{x^r}{\underline{|r}}.$$

8. Find by successive differentiation the r^{th} differential coefficient of $x^n \log x$, and show that, if r is less than n, it can be put in the form

$$\frac{\underline{|n}}{\underline{|n-r}}\, x^{n-r}\left\{\log x + \sigma\left(\frac{1}{n}\right) - \sigma\left(\frac{1}{n-r}\right)\right\}$$

if $\sigma\left(\frac{1}{n}\right)$ is used to denote $1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}$.

9. Write down the n^{th} differential coefficient of $x^n \log x$.

10. Form the successive integrals of $x^n \log x$, and show that

$$\mathrm{D}^{-r}(x^n \log x) = \frac{\underline{|n}}{\underline{|n+r}}\, x^{n+r}\left\{\log x + \sigma\left(\frac{1}{n}\right) - \sigma\left(\frac{1}{n+r}\right)\right\}$$

which is of the same form as that of $\mathrm{D}^r(x^n \log x)$, with the sign of r changed.

11. Show that the limiting value of $\frac{1}{h}\left(\frac{\underline{|n+h}}{\underline{|h}} - \underline{|n}\right)$, when $h=0$, is $\underline{|n}.\sigma\left(\frac{1}{n}\right)$.

12. Find the value of $\mathrm{D}^n(x^n \log x)$ by using the algebraic expression for $\log x$, viz. $\log x = \frac{x^h-1}{h}$ when $h=0$.

126. General formulae for the successive differential coefficients and integrals of certain other important functions can also be obtained; $\sin x$, $\cos x$, $\sin mx$, $\cos mx$, e^x, e^{ax} are among the simplest.

Thus, if $y = \sin x$,

$$\mathrm{D}y = \cos x = \sin\left(x + \frac{\pi}{2}\right),$$

$$\mathrm{D}^2 y = \cos\left(x + \frac{\pi}{2}\right) = \sin\left(x + \frac{2\pi}{2}\right),$$

....

$$\mathrm{D}^r y = \sin\left(x + \frac{r\pi}{2}\right).$$

This gives for the integrals of $\sin x$,

$$D^{-1}\sin x = \sin\left(x - \frac{\pi}{2}\right) + \text{a constant.}$$

……………………………………

$$D^{-s}\sin x = \sin\left(x - \frac{s\pi}{2}\right) + D^{-s}(0).$$

EXAMPLES.

1. Show that $D^r \cos x = \cos\left(x + \frac{r\pi}{2}\right)$, and that

$$D^{-s}\cos x = \cos\left(x - \frac{s\pi}{2}\right) + D^{-s}(0).$$

2. Show that

$$D^r \sin mx = m^r \sin\left(mx + \frac{r\pi}{2}\right),$$

and verify, by differentiation, that

$$D^{-1}\sin mx = m^{-1}\sin\left(mx - \frac{\pi}{2}\right) + D^{-1}(0)$$

$$= -\frac{1}{m}\cos mx + a, \text{ where } a \text{ is any constant.}$$

3. Show that $D^r \cos mx = m^r \cos\left(mx + \frac{r\pi}{2}\right)$, and deduce its first integral.

4. Prove that

(1) $D^r e^x = e^x$.
(2) $D^r e^{ax} = a^r e^{ax}$.
(3) $D^{-s} e^{ax} = a^{-s} e^{ax} + D^{-s}(0)$.

127. *To find the successive differential coefficients of $\tan x$.*

The differential coefficients of $\tan x$ become very complicated unless the work is done in a very systematic manner. It is not possible to obtain a general formula for the n^{th} differential coefficient. The best way of forming the successive differential coefficients seems to be as follows:

Let $$y = \tan x;$$

$$\therefore\ Dy = \sec^2 x = 1 + y^2.$$

(This value of Dy will be continually substituted in the succeeding work.)

$$
\begin{aligned}
\mathrm{D}^2y &= 2y\,\mathrm{D}y = 2y + 2y^3,\\
\mathrm{D}^3y &= (2 + 6y^2)\,\mathrm{D}y = 2 + 8y^2 + 6y^4,\\
\mathrm{D}^4y &= (16y + 24y^3)\,\mathrm{D}y\\
&= 16y + 40y^3 + 24y^5,
\end{aligned}
$$

and so on.

The work in this way can be done quite rapidly, but each stage must be gone through. We cannot write down D^ny without differentiating n times. It will be noticed that all the even orders of differential coefficients have y (*i.e.* $\tan x$) as a factor, whereas all the odd orders begin with a constant.

The integral of $\tan x$ is $\log\sec x$, as can be seen at once by differentiation.

128. *To find the successive differential coefficients of* $\sec x$.

The differential coefficients of $\sec x$ also become complicated unless we work very systematically. We shall be able to express them in terms of $\sec x$ and powers of $\tan x$. To abbreviate these expressions take

$$z = \sec x, \text{ and } y = \tan x\,;$$

$$\therefore\ \mathrm{D}z = \sec x\tan x = zy, \text{ and } \mathrm{D}y = \sec^2 x = 1 + y^2.$$

Hence

$$
\begin{aligned}
\mathrm{D}^2z &= z\,\mathrm{D}y + y\,\mathrm{D}z\\
&= z(1 + y^2) + y^2z\\
&= z(1 + 2y^2),\\
\mathrm{D}^3z &= z.4y\,\mathrm{D}y + (1 + 2y^2)\,\mathrm{D}z\\
&= z\{4y(1 + y^2) + y(1 + 2y^2)\}\\
&= z(5y + 6y^3),\\
\mathrm{D}^4z &= z(5 + 18y^2)\,\mathrm{D}y + (5y + 6y^3)\,\mathrm{D}z\\
&= z\{(5 + 23y^2 + 18y^4) + (5y^2 + 6y^4)\}\\
&= z\{5 + 28y^2 + 24y^4\},\\
&\text{etc.}
\end{aligned}
$$

We cannot obtain a general formula, but we can rapidly obtain any differential coefficient by the above method. It will be seen that $\sec x$ is a factor of all the differential coefficients, and $\tan x$ is a factor of those of odd order.

129. *To find the r^{th} differential coefficient of $e^{ax}\sin bx$.*

The function $e^{ax}\sin bx$ is an important function in connection with electrical theory.

We can find a general expression for the r^{th} differential coefficient of this function, which, by putting $r=-1$, will give us its integral.

Thus, if
$$y=e^{ax}\sin bx,$$
$$\text{D}y=e^{ax}(a\sin bx+b\cos bx).$$

Let α be an auxiliary angle, such that $\frac{b}{a}=\tan\alpha$,

so that
$$\sin\alpha=\frac{b}{\sqrt{(a^2+b^2)}},\quad \cos\alpha=\frac{a}{\sqrt{(a^2+b^2)}}.$$
$$\therefore\ \text{D}y=\sqrt{(a^2+b^2)}\,e^{ax}\sin(bx+\alpha)$$
$$=k\,e^{ax}\sin(bx+\alpha),\ \text{where } k=\sqrt{(a^2+b^2)}.$$

Similarly,
$$\text{D}^2y=k^2\,e^{ax}\sin(bx+2\alpha),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\text{D}^ry=k^r\,e^{ax}\sin(bx+r\alpha)$$
$$=(a^2+b^2)^{\frac{r}{2}}\,e^{ax}\sin\left(bx+r\tan^{-1}\frac{b}{a}\right).$$

This gives, as a particular case,
$$\text{D}^{-1}y=\frac{1}{\sqrt{(a^2+b^2)}}\,e^{ax}\sin\left(bx-\tan^{-1}\frac{b}{a}\right)$$
which is easily verified.

EXAMPLES.

1. Write down the r^{th} differential coefficient of $e^{4x}\sin 3x$.

2. Show that $\text{D}^r e^{ax}\cos bx=k^r e^{ax}\cos(bx+r\alpha)$ where $k=\sqrt{(a^2+b^2)}$, and $\tan\alpha=b\div a$.

3. Find the r^{th} differential coefficient of $e^{-ax}\sin bx$, and deduce its first integral.

4. Find the r^{th} differential coefficient of $e^{-ax}\cos bx$, and deduce its first integral. Verify this last result by differentiation.

5. Deduce the formulae for $\text{D}^r e^{ax}$ and $\text{D}^r e^{-ax}$ by putting $b=0$ in the formulae for $\text{D}^r e^{\pm ax}\cos bx$.

6. Deduce the formula for $\text{D}^r\sin bx$ by putting $a=0$ in either of the formulae for $\text{D}^r e^{\pm ax}\sin bx$.

130. Leibnitz' theorem.

We have seen how to form the successive differential coefficients of various simple functions, and other examples will be found in the next chapter, which is devoted to methods of dealing with rational algebraic fractions. We will now establish a theorem by means of which the n^{th} differential coefficient of the product of two functions of a variable x can be written down provided we know the successive differential coefficients of the separate factors. This theorem is known as *Leibnitz' theorem*.

If u and v are functions of x, we require to find a formula for $D^n(uv)$ in terms of u and v and their successive differential coefficients.

By Art. 23,

$$D(uv) = Du.v + u.Dv,$$

and, similarly,

$$\begin{aligned} D^2(uv) &= (D^2u.v + Du.Dv) + (Du.Dv + u.D^2v) \\ &= D^2u.v + 2Du.Dv + u.D^2v. \end{aligned}$$

Proceeding in the same way, we find successively

$$D^3(uv) = D^3u.v + 3D^2u.Dv + 3Du.D^2v + u.D^3v,$$

$$D^4(uv) = D^4u.v + 4D^3u.Dv + 6D^2u.D^2v + 4Du.D^3v + u.D^4v,$$

……………………………………………………………

It will be noticed that the numerical coefficients in the above formulae are just the same as those which occur in the expansions of the successive powers of $a+b$ by the binomial theorem, viz.:

1, 1	coefficients of	$(a+b)$,
1, 2, 1	..	$(a+b)^2$,
1, 3, 3, 1	,,	$(a+b)^3$,
1, 4, 6, 4, 1	,,	$(a+b)^4$,

and it will be readily seen that this is because the terms are formed in a very similar way. Thus

$$\begin{aligned} (a+b)^2 &= (a^2+ab) + (ab+b^2), \\ &= a^2 + 2ab + b^2, \end{aligned}$$

and

$$\begin{aligned} (a+b)^3 &= (a^3 + 2a^2b + ab^2) + (a^2b + 2ab^2 + b^3) \\ &= a^3 + 3a^2b + 3ab^2 + b^3. \end{aligned}$$

This corresponds exactly to the process of finding $\mathrm{D}^3(uv)$ from the second differential coefficient, $\mathrm{D}^2(uv)$, provided we work as follows: we have

$$\mathrm{D}^2(uv) = \mathrm{D}^2u.v + 2\mathrm{D}u.\mathrm{D}v + u.\mathrm{D}^2v;$$

to find $\mathrm{D}^3(uv)$ we have to differentiate separately both factors of each term; we may, therefore, *first* differentiate each of the factors D^2u, $\mathrm{D}u$, u, and *then* each of the remaining factors v, $\mathrm{D}v$, D^2v, the result being

$$\begin{aligned}\mathrm{D}^3(uv) &= (\mathrm{D}^3u.v + 2\mathrm{D}^2u.\mathrm{D}v + \mathrm{D}u.\mathrm{D}^2v) \\ &\qquad + (\mathrm{D}^2u.\mathrm{D}v + 2\mathrm{D}u.\mathrm{D}^2v + u.\mathrm{D}^3v) \\ &= \mathrm{D}^3u.v + 3\mathrm{D}^2u.\mathrm{D}v + 3\mathrm{D}u.\mathrm{D}^2v + u.\mathrm{D}^3v.\end{aligned}$$

The analogy between this process and that of forming the cube of $a+b$ from $a^2+2ab+b^2$ by multiplying every term *first* by a, and *then* by b; and adding, is thus very apparent.

It will be seen that, in each process, viz., the process of finding the successive differential coefficients of uv, and the process of finding the successive powers of $a+b$, the resulting numerical coefficients in each expansion are obtained by adding together in pairs the numerical coefficients in the preceding expansion, and it is therefore not surprising that the results are the same.

Hence the n^{th} differential coefficient of uv is

$$\mathrm{D}^nu.v + n\mathrm{D}^{n-1}u.\mathrm{D}v + \frac{n(n-1)}{1.2}\mathrm{D}^{n-2}u.\mathrm{D}^2v + \dots$$
$$+ \frac{n(n-1)}{1.2}\mathrm{D}^2u.\mathrm{D}^{n-2}v + n\mathrm{D}u.\mathrm{D}^{n-1}v + u.\mathrm{D}^nv.$$

This is Leibnitz' theorem.

131. Symbolical representation of Leibnitz' theorem.

If D_1 be used to denote the process of differentiating u and its successive differential coefficients, and D_2 be used to denote the same operation performed on v and its differential coefficients, the differential coefficient of uv may be represented by $(\mathrm{D}_1 + \mathrm{D}_2)uv$;

for D_1uv will denote $\mathrm{D}u.v$ by the hypothesis,
and D_2uv „ $u.\mathrm{D}v$ „

Similarly, the expression found above for $D^2(uv)$, viz.:

$$D^2u.v + 2Du.Dv + u.D^2v,$$

may be written in the symbolic form

$$(D_1{}^2 + 2D_1D_2 + D_2{}^2)uv,$$

which may be condensed into the form

$$(D_1 + D_2)^2uv;$$

and so on, the general formula being

$$\mathbf{D^n uv = (D_1 + D_2)^n uv.}$$

132. Let us, by way of illustration, find the n^{th} differential coefficient of x^2e^x by means of Leibnitz' theorem.

Let $y = e^x.x^2$;

$$\therefore\ D^ny = D^n(e^x).x^2 + nD^{n-1}(e^x)D(x^2) + \frac{n(n-1)}{1.2}D^{n-2}(e^x)D^2(x^2) + \dots$$

$$= e^x\left\{x^2 + n.2x + \frac{n(n-1)}{1.2}.2\right\},$$

since all the differential coefficients of e^x are equal to e^x, and $Dx^2 = 2x$, $D^2(x^2) = 2$, and all the remaining differential coefficients of x^2 are equal to 0.

The result may be, of course, written

$$D^n(x^2e^x) = e^x\{x^2 + 2nx + n(n-1)\}.$$

The above is an example of the following important theorem which results from applying Leibnitz' theorem to a product in which one of the factors is e^{ax}.

133. To prove that

$$\mathbf{D^n(u.e^{ax}) = e^{ax}(D + a)^n u.}$$

If $y = u.e^{ax}$,

$$D^ny = e^{ax}.D^nu + n.ae^{ax}.D^{n-1}u + \frac{n(n-1)}{1.2}.a^2e^{ax}D^{n-2}u + \dots$$

$$= e^{ax}\left(D^nu + a.nD^{n-1}u + a^2.\frac{n(n-1)}{1.2}D^{n-2}u + \dots\right)$$

$$= e^{ax}(D + a)^nu.$$

Thus, in the above example,

$$\begin{aligned}\mathrm{D}^n(x^2e^x) &= e^x(1+\mathrm{D})^n x^2\\ &= e^x\left(x^2+n.2x+\frac{n(n-1)}{1.2}.2\right)\\ &= e^x\{x^2+2n.x+n(n-1)\},\end{aligned}$$

and, more generally,

$$\begin{aligned}\mathrm{D}^n(x^2.e^{ax}) &= e^{ax}(a+\mathrm{D})^n x^2\\ &= e^{ax}\left(a^n.x^2+na^{n-1}.2x+\frac{n(n-1)}{1.2}a^{n-2}.2\right)\\ &= a^{n-2}e^{ax}\{a^2x^2+2nax+n(n-1)\}.\end{aligned}$$

134. The formula of last article will give us the successive *integrals* of x^2e^x by putting n successively equal to -1, -2, etc.

Thus
$$\begin{aligned}\mathrm{D}^{-1}(x^2e^x) &= e^x(1+\mathrm{D})^{-1}x^2\\ &= e^x(1-\mathrm{D}+\mathrm{D}^2-\dots)x^2, \text{ by the binomial theorem,}\\ &= e^x(x^2-2x+2),\\ \mathrm{D}^{-2}(x^2e^x) &= \mathrm{D}^{-1}\{e^x(x^2-2x+2)\}\\ &= e^x(1+\mathrm{D})^{-1}(x^2-2x+2)\\ &= e^x(1-\mathrm{D}+\mathrm{D}^2-\dots)(x^2-2x+2)\\ &= e^x\{(x^2-2x+2)-(2x-2)+2\}\\ &= e^x(x^2-4x+6),\end{aligned}$$

or, more rapidly,

$$\begin{aligned}\mathrm{D}^{-2}(x^2e^x) &= e^x(1+\mathrm{D})^{-2}x^2\\ &= e^x(1-2\mathrm{D}+3\mathrm{D}^2-\dots)x^2\\ &= e^x(x^2-2.2x+3.2)\\ &= e^x(x^2-4x+6).\end{aligned}$$

The student can verify these results by successive differentiations, starting with $e^x(x^2-4x+6)$, so obtaining the series

$$\begin{aligned}y &&&= e^x(x^2-4x+6),\\ \mathrm{D}y &= e^x(x^2-4x+6+2x-4) &&= e^x(x^2-2x+2),\\ \mathrm{D}^2y &= e^x(x^2-2x+2+2x-2) &&= e^x.x^2,\\ \mathrm{D}^3y &= e^x(x^2+2x) &&= e^x(x^2+2x)\\ \mathrm{D}^4y &= e^x(x^2+2x+2x+2) &&= e^x(x^2+4x+2),\\ \mathrm{D}^5y &= e^x(x^2+4x+2+2x+4) &&= e^x(x^2+6x+6),\\ \mathrm{D}^6y &= e^x(x^2+6x+6+2x+6) &&= e^x(x^2+8x+12),\\ &\text{etc.,}\end{aligned}$$

using the formula $\mathrm{D}e^xf(x)=e^x(1+\mathrm{D})f(x)$ each time.

By using the formula $\text{D}^n e^x f(x) = e^x(1+\text{D})^n f(x)$, the student may also verify the last result as being the

6th differential coefficient of $e^x(x^2-4x+6)$,
or the 5th .. .. $e^x(x^2-2x+2)$,
.. 4th $e^x.x^2$,
etc., etc.

He will find the time well spent in using the above formula backwards and forwards for different values of n until he is certain of being able to use it without error. The formula is very valuable in connection with the solution of certain classes of differential equations.

135. Successive differential coefficients connected by a differential equation.

In some cases where we cannot actually find a formula for the n^{th} differential coefficient of a function, we may be able to establish a relation between its successive differential coefficients which may often be of service.

Thus, if
$$y = \tan x,$$
$$\text{D}y = 1 + y^2,$$
$$= 1 + y.y.$$

We can write down the result of differentiating this n times, using Leibnitz' theorem for the n^{th} differential coefficient of the product $y.y$. Thus

$$\begin{aligned}\text{D}^{n+1}y = \text{D}^n yy &= (\text{D}_1+\text{D}_2)^n yy\\ &= y\text{D}^n y + n\text{D}y\text{D}^{n-1}y + \frac{n(n-1)}{\underline{|2}}\text{D}^2y\text{D}^{n-2}y + \dots\\ &\quad + \frac{n(n-1)}{\underline{|2}}\text{D}^{n-2}y\text{D}^2y + n\text{D}^{n-1}y.\text{D}y + \text{D}^n y.y,\end{aligned}$$

the latter half of the series being a repetition of the first half, but in reverse order.

It may be interesting to take special cases, writing $y_1, y_2, \dots,$ instead of $\text{D}y$, D^2y, ..., for brevity.

Thus
$$\begin{aligned}y_2 &= yy_1 + y_1y = 2yy_1,\\ y_3 &= yy_2 + 2y_1^2 + y_2y = 2(yy_2 + y_1^2),\\ y_4 &= 2(yy_3 + 3y_1y_2),\\ y_5 &= 2(yy_4 + 4y_1y_3 + 3y_2^2),\\ &\text{etc.}\end{aligned}$$

We shall find these series useful in connection with the expansion of $\tan x$ in powers of x (Chapter XI.).

The differential coefficients of $\sec x$ can be connected with each other and with those of $\tan x$ in a similar manner.

Let $z = \sec x$ and $y = \tan x$,

then
$$z_1 = \tan x \sec x = yz.$$

Hence, by Leibnitz' theorem,
$$z_{n+1} = y_nz + ny_{n-1}z_1 + \dots + ny_1z_{n-1} + yz_n.$$

In these two examples all the differential coefficients up to the n^{th} are required in the expression for the $n+1^{\text{th}}$ differential coefficient, but in the case of many functions a much simpler relation can be found.

Thus, if
$$y = \sin^{-1}x,$$
$$y_1 = \frac{1}{\sqrt{(1-x^2)}};$$
$$\therefore\ (1-x^2)y_1^2 = 1.$$

Differentiating this, we obtain
$$2(1-x^2)y_1y_2 - 2xy_1^2 = 0,$$
$$\textit{i.e.},\ (1-x^2)y_2 = xy_1.$$

Hence, applying Leibnitz' theorem to both sides, we obtain
$$(1-x^2)y_{n+2} + n(-2x)y_{n+1} + \frac{n(n-1)}{1.2}(-2)y_n = xy_{n+1} + ny_n\ ;$$
$$\therefore\ (1-x^2)y_{n+2} = (2n+1)xy_{n+1} + n^2y_n.$$

This series will also be referred to again in Chapter XI.

EXAMPLES.

1. Prove by actual differentiation of the identity

$$\mathrm{D}^n(uv)=\mathrm{D}^n u.v+n\mathrm{D}^{n-1}u.\mathrm{D}v+\frac{n(n-1)}{1.2}\mathrm{D}^{n-2}u.\mathrm{D}^2v+\dots$$

that $\mathrm{D}^{n+1}(uv)=\mathrm{D}^{n+1}u.v+(n+1)\mathrm{D}^n u.\mathrm{D}v+\frac{(n+1)n}{1.2}\mathrm{D}^{n-1}u.\mathrm{D}^2v+\dots.$

[NOTE.—The proof of this will establish the fact that if Leibnitz' theorem holds for n differentiations it will hold for $n+1$ differentiations, and therefore universally.]

2. Find the n^{th} differential coefficient of $x\log x$; also of $x^2\log x$.

3. Find the 3^{rd} integral of $x\log x$. [Use the formula

$$(\mathrm{D}_2+\mathrm{D}_1)^{-3}(x\log x)=(\mathrm{D}_2^{-3}-3\mathrm{D}_1\mathrm{D}_2^{-4})\,x\log x,\ \text{etc.}]$$

4. If $y=e^{-3x}(x^2-3x+4)$, find D^5y and $\mathrm{D}^{-3}y$. Verify by deducing D^5y *directly* from the expression found for $\mathrm{D}^{-3}y$, and *vice versa*.

5. If $y=x^2\sin 2x$, find D^5y and $\mathrm{D}^{-3}y$.

6. If $y=\sin(m\sin^{-1}x)$, prove that $(1-x^2)y_2-xy_1+m^2y=0$.

7. Find the result of differentiating the above differential equation n times.

8. If $y=e^{ax}(\mathrm{A}\sin bx+\mathrm{B}\cos bx)$, prove that

$$y_2-2ay_1+(a^2+b^2)y=0.$$

9. If $y=e^{a\sin^{-1}x}$, prove that $(1-x^2)y_2=xy_1+a^2y$, and find the result of differentiating this relation n times.

10. If $y=\tan x$, prove that

$$y_n\cos x+ny_{n-1}\cos\left(x+\frac{\pi}{2}\right)+\frac{n(n-1)}{1.2}y_{n-2}\cos\left(x+\frac{2\pi}{2}\right)+\dots$$
$$=\sin\left(x+\frac{n\pi}{2}\right).$$

11. If $y=\sec x$, show that the same expression as in No. 10, viz. $y_n\cos x+\text{etc.}$, is equal to zero.

12. Obtain similar formulæ in the case of (1) $y=\cot x$, (2) $y=\operatorname{cosec} x$.

CHAPTER X.

SUCCESSIVE DIFFERENTIATION—*Continued.* RATIONAL FRACTIONS—PARTIAL FRACTIONS.

136. Rational Algebraic Fractions. Proper Fractions. In the preceding chapter it has been shown how to find the successive differential coefficients, and the integrals, of certain simple functions. Another very important class of functions, which are called Rational Algebraic Fractions, will be treated in the present chapter. The simplest form of such fraction is $\dfrac{A}{ax+b}$, where the numerator is a constant and the denominator is an algebraic function of x of the first degree.

In general a rational algebraic fraction is a fraction whose numerator and denominator are rational algebraic functions of x (or some other variable), *i.e.* are expressions all of whose terms are multiples of integer powers of x.

Thus $\dfrac{x^3-2x^2+3x-1}{(x-1)(x^2+x+1)}$, and $\dfrac{x^4}{2x^3-7}$, and $\dfrac{\dfrac{x^2}{x-1}}{3+\dfrac{2x}{x+1}}$ are all examples of rational fractions. The last is a complex fraction, but any complex rational fraction can easily be reduced to a non-complex form; also any rational fraction can be reduced to its lowest terms by cancelling common factors between the numerator and denominator. Moreover, if the degree of the numerator is as high as, or higher than that of the denominator, the fraction can be expressed in the form of a Quotient + a Remainder whose numerator is of lower degree than the denominator, just as an arithmetical improper fraction can be reduced to the sum of a whole number and a proper fraction. In the greater part of what follows we will suppose that this has been done, so that the fractions we shall deal with will be in their lowest terms, and will have their numerators of lower degree than their denominators. Such fractions will be called Proper Fractions.

Example. Reduce each of the above three fractions to the form of *quotient + proper fraction.*

137. Simple Fractions. The simplest form of proper fraction is one in which the denominator is of the first degree. Thus

$$\frac{1}{ax+b} \text{ is a simple fraction.}$$

It is easy to write down the n^{th} differential coefficient of such a fraction. For, let

$$\begin{aligned} y &= (ax+b)^{-1}; \\ \therefore\ \mathrm{D}y &= -1.(ax+b)^{-2}\mathrm{D}(ax+b) \\ &= -a(ax+b)^{-2}. \end{aligned}$$

Similarly,
$$\mathrm{D}^2y = (-a)(-2a)(ax+b)^{-3},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\begin{aligned} \mathrm{D}^ny &= (-a)(-2a)\ldots(-na)(ax+b)^{-n-1} \\ &= \frac{(-a)^n \underline{|n}}{(ax+b)^{n+1}}. \end{aligned}$$

The first integral of $\dfrac{1}{ax+b}$ is $\dfrac{1}{a}\log(ax+b)$, as can easily be verified by differentiating $\log(ax+b)$.

Next in order of simplicity come such fractions as $\dfrac{1}{(ax+b)^2}$, $\dfrac{1}{(ax+b)^3}$, ... The differentiation and integration of these present no new difficulties. The numerators need not be unity; all that is needful is that they shall be constants.

Other fractions which present more difficulty, but which must be considered as simple fractions, are

$$\frac{\mathrm{A}x+\mathrm{B}}{(x+a)^2+b^2},\ \frac{\mathrm{A}x+\mathrm{B}}{\{(x+a)^2+b^2\}^2},\ldots$$

the peculiarity of these fractions being that the quadratic expressions in the denominators are *irreducible*, *i.e.* cannot be factorized into real factors, and that the numerators are of the first degree. It will be found that all proper fractions can be expressed as the sum of such simple fractions as are indicated in this article.

EXAMPLES.

1. Find the n^{th} differential coefficients, and the first integrals of

$$\frac{1}{(ax+b)^r} \text{ and } \frac{1}{(b-ax)^r}.$$

2. Write down the first integrals of

$$\frac{1}{2x-3}, \quad \frac{3}{(2x-3)^3}, \quad \frac{2}{2-5x}, \quad \frac{4}{(2-5x)^2}.$$

3. State which of the following fractions are simple fractions:

$$\frac{3}{x^2+2x}, \quad \frac{3}{x^2+2x+1}, \quad \frac{3}{x^2+2x+2}.$$

4. Show that $\dfrac{x+3}{x^2+2x+1}$ can be put into the form $\dfrac{1}{(x+1)}+\dfrac{2}{(x+1)^2}$.

5. Show that $\dfrac{x^3+2x^2+3x+4}{(x^2+4)^2}$ can be put into the form $\dfrac{x+2}{x^2+4}-\dfrac{x+4}{(x^2+4)^2}$.

[Divide the numerator by x^2+4; it will be found that the quotient is $x+2$ and that the remainder is $-x-4$.]

138. Partial Fractions. To succeed in obtaining the n^{th} differential coefficient or the integral of a proper fraction which is not one of the simple fractions indicated in Art. 137, the fraction must, in general, be broken up into the sum or difference of simple fractions whose denominators will be the several factors of the given denominator. This process is the opposite of the process of adding together a number of simple fractions into one fraction, and like most reverse operations it is not so easy as the direct process. The greater part of this chapter will be devoted to explaining methods of doing this work. For purposes of integration such breaking up into *partial fractions* is essential, and it is important that the student should acquire some facility in the work. The idea is simple enough, but the details are in some cases troublesome. The various methods of saving labour in the work will be explained by means of examples. We will start with one simple example and then explain the general theory.

Example. Find the integral, and the n^{th} differential coefficient, of

$$\frac{3}{x^2-4}.$$

The given fraction can be expressed in the form

$$\frac{3}{4}\left(\frac{1}{x-2}-\frac{1}{x+2}\right);$$

$\therefore$ its first integral is

$$\tfrac{3}{4}\{\log(x-2)-\log(x+2)\},$$

and its n^{th} differential coefficient

$$=\frac{3}{4}(-1)^n \underline{|n}\left\{\frac{1}{(x-2)^{n+1}}-\frac{1}{(x+2)^{n+1}}\right\}.$$

We proceed to consider the various cases which may arise in the treatment of any proper fraction.

If the denominator is of the first degree, the fraction is already in its simplest form; *e.g.*

$$\frac{1}{x+3},\ \frac{2}{3x+4},\ \dots.$$

If the denominator is of degree higher than the first, it can, in general, be broken up into factors. Several cases will arise, according to whether the denominator can be wholly broken up into real factors of the first degree or not, and whether any of its factors are repeated or not. For example, the denominators $(x-1)(x-2)(x+3)$, $(x-2)^2(x+3)$, x^2+x+1, $(x^2+x+1)^3$ are all of essentially distinct forms, and will require somewhat different treatment.

139. Unrepeated linear factors in the denominator. The first case which presents itself is that of a fraction whose denominator can be entirely broken up into real factors of the first degree, all different. We have to show how to find the simple fractions whose sum will be equal to the given fraction. Their denominators must, of course, be the several factors of the given denominator: the problem is how to find their numerators. The best way of showing the various methods of doing this will be by means of an example.

Thus
$$\frac{2x^2-3}{(x-1)(x-2)(x+3)}$$
can be made up of the sum of
$$\frac{A}{x-1}+\frac{B}{x-2}+\frac{C}{x+3},$$
where A, B, C are constants whose values have to be such that when the three simple fractions are added together, the numerator shall be $2x^2-3$.

Now, if the fractions are added together, the resulting numerator is
$$\begin{aligned}&A(x-2)(x+3)+B(x-1)(x+3)+C(x-1)(x-2)\\&\qquad=(A+B+C)x^2+(A+2B-3C)x-(6A+3B-2C).\end{aligned}$$

If this is to become $2x^2-3$, we must have
$$\left.\begin{aligned}A+B+C&=2\\A+2B-3C&=0\\6A+3B-2C&=3\end{aligned}\right\}\quad\therefore\ A=\tfrac{1}{4},\ B=1,\ C=\tfrac{3}{4};$$
$$\therefore\ \frac{2x^2-3}{(x-1)(x-2)(x+3)}=\frac{1}{4}\left(\frac{1}{x-1}+\frac{4}{x-2}+\frac{3}{x+3}\right).$$

Therefore its n^{th} differential coefficient
$$=\frac{(-1)^n\lfloor n}{4}\left\{\frac{1}{(x-1)^{n+1}}+\frac{4}{(x-2)^{n+1}}+\frac{3}{(x+3)^{n+1}}\right\},$$
and its integral is
$$\tfrac{1}{4}\{\log(x-1)+4\log(x-2)+3\log(x+3)\},$$
as can be at once verified by differentiation.

It will be seen by careful study of the above example that while there will always be as many constants as there are simple fractions, so that the number of constants is equal to the degree of the original denominator, the degree of the resulting numerator when the fractions are added together will be one less than the degree of the denominator. Consequently the number of constants is just sufficient to satisfy all the conditions required to make this numerator identical with the given numerator. Thus in the above example there were three constants and three conditions to be satisfied.

The method adopted above for finding the values of the constants is easy to understand, but it is not usually the simplest method to use practically. A better way in practice will be now shown. We had, on comparing numerators, to make

$$\text{A}(x-2)(x+3)+\text{B}(x-1)(x+3)+\text{C}(x-1)(x-2)=2x^2-3$$

identically. ....(N)

Now, if the two sides of (N) are identically equal they will be equal whatever value we please to give to x. It is easy to choose special values of x which will enable us to determine A, B, C separately instead of by solving three simultaneous equations. Thus if $x=1$, the multipliers of B and C will be zero, and we are left with $-4\text{A}=-1$, whence $\text{A}=\frac{1}{4}$.

And if $x=2$, we find that $5\text{B}=5$; $\therefore$ $\text{B}=1$.

Finally, if $x=-3$, we find that $20\text{C}=15$; $\therefore$ $\text{C}=\frac{3}{4}$.

We ought to verify the results we have obtained, either by giving another value to x, as for example $x=0$, or by comparing the coefficients of the different powers of x on each side of (N). If these agree our work is right.

Another mode of finding the constants will be found interesting and may be used when desired instead of either of the above methods.

Let us consider one factor of the denominator at a time. Suppose the given fraction is $\dfrac{f(x)}{(x-a)\phi(x)}$ and that it is broken up into the simple fraction $\dfrac{\text{A}}{x-a}+$ a number of others whose sum will be of the form $\dfrac{u}{\phi(x)}$.

Then
$$\frac{f(x)}{(x-a)\phi(x)}=\frac{\text{A}}{x-a}+\frac{u}{\phi(x)};$$

and, comparing numerators, we have the identity

$$f(x)=\text{A}\phi(x)+u(x-a). \;\ldots\ldots\ldots\ldots\ldots\ldots\text{(N)}$$

In this identity put $x=a$, and we find

$$f\,a)=\text{A}.\phi(a);$$

$$\therefore\quad \text{A}=\frac{f(a)}{\phi(a)},$$

and therefore the partial fraction

$$\frac{\text{A}}{x-a}=\frac{f(a)}{(x-a)\,\phi(a)},$$

i.e. the partial fraction corresponding to $x-a$ is obtainable immediately from the original fraction by the simple plan of putting $x=a$ in every part of the fraction except the factor $x-a$. We will, however, deduce another formula for A which is sometimes slightly more convenient. Let the original denominator be $\text{F}(x)$, so that

$$\text{F}(x)=(x-a)\,\phi(x).$$

Hence $$\text{F}'(x)=\phi(x)+(x-a)\,\phi'(x).$$

Now put $x=a$ in this identity, and we obtain

$$\text{F}'(a)=\phi(a).$$

Hence $$\text{A}=\frac{f(a)}{\text{F}'(a)}.$$

Applying this to the given example, viz.

$$\frac{2x^2-3}{(x-1)(x-2)(x+3)},$$

the values of a corresponding to each of the three simple fractions into which it can be split are 1, 2, -3 respectively.

Also $f(x)=2x^2-3$; $\therefore$ $f(1)=-1$, $f(2)=5$, $f(-3)=15$,
and $\text{F}(x)=x^3-7x+6$ (as seen by multiplying out);
$\therefore$ $\text{F}'(x)=3x^2-7$; and $\text{F}'(1)=-4$, $\text{F}'(2)=5$, $\text{F}'(-3)=20$;
$\therefore$ $\text{A}=\frac{1}{4}$, $\text{B}=1$, $\text{C}=\frac{3}{4}$ as before.

NOTE.—It must be remembered that the above method has been established on the supposition that the simple factors of $\text{F}(x)$ are not repeated. If a factor is repeated, the method will fail in the case of that factor, but will still be applicable to such of the factors as are not repeated.

The method is valid whether the given fraction is proper or improper, but it must be remembered that in the latter case there is a quotient in addition to the sum of the simple fractions.

140. Repeated linear factors in the denominator.—We will now consider the case of a repeated factor in the denominator.

Take first $$\frac{2x^2-3x+4}{x^3}.$$

This can evidently be broken up into

$$\frac{2}{x}-\frac{3}{x^2}+\frac{4}{x^3}.$$

Similarly, if we have

$$\frac{2x^2-3x+4}{(x-1)^3},$$

we may break it up into

$$\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{(x-1)^3},$$

where A, B, C are constants to be determined.

Adding the fractions together, and equating the numerator so obtained to the given numerator, we have

$$A(x-1)^2+B(x-1)+C=2x^2-3x+4$$

identically. ....(N)

The constants A, B, C are just sufficient in number to enable us to satisfy all the necessary conditions. Several ways of finding the values of A, B, C may be adopted. Thus we may equate the coefficients of the different powers of x, obtaining three simultaneous equations. The solution by this method is left to the student.

Or, we may try the effect of putting $x=1$ in the above identity. This gives us $C=3$.

We may now, if we like, transfer C to the other side, and divide both sides by $x-1$.

This gives us $A(x-1)+B=2x-1$ identically; whence, by putting $x=1$, we find $B=1$; and, equating the coefficients of x on the two sides, we obtain $A=2$.

Or, again, having found $C=3$, we may differentiate both sides of (N), obtaining

$$2A(x-1)+B=4x-3;$$

whence, by putting $x=1$, we find $B=1$, and lastly $A=2$ as before.

Another method may be shown by which all the constants can be determined simply.

Thus, taking the same fraction

$$\frac{2x^2-3x+4}{(x-1)^3},$$

let $$x-1=y;$$

then $$2x^2-3x+4=2(y+1)^2-3(y+1)+4$$
$$=2y^2+y+3;$$

$$\therefore\ \text{the fraction}=\frac{2y^2+y+3}{y^3}=\frac{2}{y}+\frac{1}{y^2}+\frac{3}{y^3},$$

which becomes, on replacing y by its equivalent $x-1$,

$$\frac{2}{x-1}+\frac{1}{(x-1)^2}+\frac{3}{(x-1)^3}.$$

This method is very powerful even in cases where the denominator has other factors besides the repeated factor. Thus, consider the fraction

$$\frac{3x^3-5x^2+4}{(x-1)^3(x^2+1)}.$$

$$\begin{array}{r} 3+9+\ \ 9+3 \\ -5-10-5 \\ +4 \\ \hline 3+4-\ \ 1+2 \end{array}$$

Putting $x-1=y$, the fraction reduces to

$$\frac{3y^3+4y^2-y+2}{y^3(y^2+2y+2)}.$$

If now we divide the numerator by $2+2y+y^2$ so as to obtain the quotient in *ascending* powers of y, till the remainder is divisible by y^3, we shall have completely determined the partial fractions required,

$$\begin{array}{r|l} 2+2y+y^2 & 2-\ y\ +4y^2+3y^3\ \big(\ \underline{1-\tfrac{3}{2}y+3y^2} \\ & \underline{2+2y\ +\ y^2\qquad\quad} \\ & -3y\ +3y^2+3y^3 \\ & \underline{-3y\ -3y^2-\tfrac{3}{2}y^3} \\ & 6y^2+\tfrac{9}{2}y^3\qquad \\ & \underline{6y^2+6y^3+3y^4} \\ & -\tfrac{3}{2}y^3-3y^4 \end{array}$$

viz. $$\frac{1}{y^3}-\frac{3}{2y^2}+\frac{3}{y}-\frac{\frac{3}{2}+3y}{y^2+2y+2},$$

$$=\frac{1}{(x-1)^3}-\frac{3}{2(x-1)^2}+\frac{3}{x-1}+\frac{\frac{3}{2}-3x}{x^2+1}.$$

NOTE.—The last of the partial fractions has a quadratic denominator which cannot be split into real factors. We shall consider this kind of fraction more particularly in Article 142. Meanwhile all that is necessary to observe is that it is a proper fraction with a first degree numerator, its denominator being of the second degree and irreducible. It belongs therefore to the class of simple fractions (see Art. 137).

141. Method of procedure in the case of improper fractions. If the fraction is an improper fraction, *i.e.* has a numerator of degree equal to or greater than that of the denominator, it must be put in the form of Quotient + Remainder, as stated at the beginning of the chapter. This work can often be conveniently performed without actual division in the following manner.

Let the fraction be $\dfrac{3x^3-5x^2+7x+2}{x^2-3x+2}$.

The quotient will evidently be of the first degree; call it therefore $\text{A}x+\text{B}$, where A and B are constants to be determined. The remainder will be a proper fraction, and is required in the form $\dfrac{\text{C}}{x-1}+\dfrac{\text{D}}{x-2}$. We therefore assume the given fraction to be equal to $\text{A}x+\text{B}+\dfrac{\text{C}}{x-1}+\dfrac{\text{D}}{x-2}$.

The identical relation from which A, B, C, and D are to be determined is obtained by adding the assumed equivalent expressions into one fraction and identifying its numerator with the given numerator. The relation is, therefore,

$$(\text{A}x+\text{B})(x-1)(x-2)+\text{C}(x-2)+\text{D}(x-1)$$
$$=3x^3-5x^2+7x+2. \quad \ldots\ldots\ldots \text{(N)}$$

By comparing the coefficients of x^3, we see that $\text{A}=3$.*

.. „ „ x^2, „ $\text{B}-3\text{A}=-5$, $\therefore$ $\text{B}=4$.

By putting $x=1$, „ $\text{C}=-7$.

„ $x=2$, „ $\text{D}=20$.

---

* It was obvious in the first instance that the first term of the quotient must be $3x$, since it is obtained by dividing $3x^3$ by x^2. We might, therefore, have assumed the quotient to be $3x+\text{B}$ instead of $\text{A}x+\text{B}$.

$$\therefore \text{ the given fraction} = 3x + 4 - \frac{7}{x-1} + \frac{20}{x-2}.$$

Alternative method. Make the left-hand side of (N) divisible by $x-2$ by subtracting D from both sides, and determine the value of D from the principle that $x-2$ must exactly divide the right-hand side also, since both sides are identically equal. This condition will give us $\text{D} = 20$, and the result of the division will leave us with the new identity

$$(\text{A}x+\text{B})(x-1)+\text{C}+20 = 3x^2+x+9\,;$$

$$\therefore\ (\text{A}x+\text{B})(x-1) = 3x^2+x-(11+\text{C}).$$

Now divide by $x-1$, and we find $\text{C} = -7$, and

$$\text{A}x+\text{B} = 3x+4.$$

The student might practise this division method by adding C to both sides of (N) and dividing through by $x-1$.

These various methods have been given to show that there are almost innumerable methods of working, all depending on the fact that we are dealing with an identity, and that whatever is true of one side of an identity is necessarily true of the other.

EXAMPLES.

Reduce the following expressions to a series of partial fractions, and write down their first integrals where possible :

1. $\dfrac{3x-1}{x^2+x-6}$. **2.** $\dfrac{2x+5}{x^2-x-2}$.

3. $\dfrac{x^2-2}{(x+2)(x+1)^2}$. **4.** $\dfrac{5x+3}{x^3-9x}$.

5. $\dfrac{(b-c)(c-a)(a-b)}{(x-a)(x-b)(x-c)}$. **6.** $\dfrac{4x^2}{(2x-3)^3}$.

7. $\dfrac{x^4-3x+7}{(x-4)(x+3)}$ $\left[\text{Assume the fraction} = x^2+\text{B}x+\text{C}+\dfrac{\text{D}}{x-4}+\dfrac{\text{E}}{x+3}\right]$.

8. $\dfrac{3x^2-5x+2}{(x+2)^2(x-3)^2}$. **9.** $\dfrac{2x^2-3x+4}{(x-1)^3(x+2)}$.

10. $\dfrac{4x-5}{(x-2)^3(x^2+x+3)}$. **11.** $\dfrac{4x^3-3x+2}{(x-2)^3(x^2+3x+2)}$. **12.** $\dfrac{13}{(x^2+1)(2x+3)}$.

13. Show that $\dfrac{1.2\ldots n}{x(x+1)\ldots(x+n)} = \dfrac{\text{C}_0}{x} - \dfrac{\text{C}_1}{x+1} + \dfrac{\text{C}_2}{x+2} - \ldots + (-1)^n\dfrac{\text{C}_n}{x+n}$,

where $\text{C}_r \equiv$ number of combinations of n things taken r together.

142. Irreducible quadratic factors in the denominator. In the previous investigations, methods have been given for facilitating the work of replacing a fraction by a series of equivalent simple fractions when the denominator of the given fraction can be split into factors of the form $(x-a)^r$. The simple fractions corresponding to each such factor are of the form

$$\frac{A}{(x-a)}+\frac{B}{(x-a)^2}+\dots+\frac{K}{(x-a)^r}.$$

In addition to this we have seen how in some cases to proceed when one of the factors of the denominator was of the second degree, and was not able to be split into two real first degree factors. The general type of such a quadratic factor is $(x+a)^2+b^2$. In such cases the corresponding partial fraction has a numerator of the form $Ax+B$. $\left(\text{See Art. 137, and the last example of Art. 140, where one of the partial fractions was } \frac{\frac{3}{2}-3x}{x^2+1}\right)$. In the case referred to, the numerator was found incidentally as it were, in the process of finding the other partial fractions. It is not always possible to proceed in that way, however, and it is desirable to consider other methods of finding the numerators in such cases. These methods we shall show by taking special examples.

Take for example the fraction $\dfrac{3x^2-5x+1}{(x^2+1)(x^2+4)}$.

This can be broken up into two fractions of the form

$$\frac{Ax+B}{x^2+1}+\frac{Cx+D}{x^2+4},$$

where the four constants A, B, C and D, are to be found by adding the assumed fractions together, and identifying the numerator so obtained with the numerator of the given fraction. The identity required is thus

$$(Ax+B)(x^2+4)+(Cx+D)(x^2+1)\equiv{}^{*}3x^2-5x+1\dots(\text{N}).$$

---

* This useful symbol is often used to signify an identity, *i.e.* an equation in which the two sides are to be equal absolutely, whatever value x may have. We might have used it in the previous articles.

Method of equating coefficients. The fundamental method of finding the constants is to equate the coefficients of the different powers of x on each side of (N). Thus

$$\begin{array}{lll} \textit{Coefficient of } x^3 & \ldots\ldots\ldots\ldots & \mathrm{A}+\mathrm{C}=0. \\ \quad ,, \quad x^2 & \ldots\ldots\ldots\ldots & \mathrm{B}+\mathrm{D}=3. \\ \quad ,, \quad x & \ldots\ldots\ldots\ldots & 4\mathrm{A}+\mathrm{C}=-5. \\ \textit{Constant term} & \ldots\ldots\ldots\ldots & 4\mathrm{B}+\mathrm{D}=1. \end{array}$$

These four equations are just sufficient to determine the constants. On solving them, we find

$$\mathrm{A}=-\frac{5}{3},\quad \mathrm{B}=-\frac{2}{3},\quad \mathrm{C}=\frac{5}{3},\quad \mathrm{D}=\frac{11}{3}.$$

We may, however, just as in the previous articles, adopt other methods of finding the values of these constants, all depending on the above identical equation between the numerator of the given fraction and its assumed equivalent, which we have denoted by (N).

Method of forced division. If $3(\mathrm{C}x+\mathrm{D})$ is added to the left-hand side of (N), it will become divisible by x^2+4, the quotient being $\mathrm{A}x+\mathrm{B}+\mathrm{C}x+\mathrm{D}$; and if the same quantity is added to the right-hand side to preserve the identity, it must also be divisible by x^2+4. The modified identity is

$$\begin{aligned}(\mathrm{A}x+\mathrm{B}+\mathrm{C}x+\mathrm{D})(x^2+4) &\equiv 3x^2-5x+1+3(\mathrm{C}x+\mathrm{D}) \\ &\equiv 3x^2+(3\mathrm{C}-5)x+(3\mathrm{D}+1).\end{aligned}$$

Now, since the right-hand side must be a multiple of x^2+4, and its first term is $3x^2$, it must obviously be $3(x^2+4)$, *i.e.* $3x^2+12$, hence

$$\left.\begin{aligned} 3\mathrm{C}-5&=0 \\ 3\mathrm{D}+1&=12 \end{aligned}\right| \qquad \therefore\ \mathrm{C}=\frac{5}{3},\quad \mathrm{D}=\frac{11}{3};$$

and, on performing the division, we have the identity

$$\mathrm{A}x+\mathrm{B}+\mathrm{C}x+\mathrm{D}\equiv 3;$$

$$\left.\begin{aligned} \therefore\ \mathrm{A}+\mathrm{C}&=0 \\ \mathrm{B}+\mathrm{D}&=3 \end{aligned}\right| \qquad \therefore\ \mathrm{A}=-\frac{5}{3},\quad \mathrm{B}=-\frac{2}{3}.$$

This method has taken some time to explain, but it is very rapid in practice, and it is well worth a little trouble to get used to it.

The student might, by way of exercise, make the original identity (N) divisible by x^2+1, instead of by x^2+4, by subtracting $3(\mathrm{A}x+\mathrm{B})$ from both sides.*

Method of substitution. If in (N) we put $x^2=-1$ (which, it must be noted, makes x imaginary), we shall be able to find A and B without being hampered with C and D, since the coefficient of $\mathrm{C}x+\mathrm{D}$ will be equal to zero. Similarly if we put $x^2=-4$, we can find C and D. (The fact that these substitutions make x imaginary is of great importance, as it enables us to find *two* coefficients at the same time.)

First, to find A and B. Since $x^2=-1$, (N) becomes

$$3(\mathrm{A}x+\mathrm{B})=-5x-2$$

$$\therefore\ \mathrm{A}x+\mathrm{B}=-\frac{5}{3}x-\frac{2}{3}.$$

Since x is imaginary, we must therefore have $\mathrm{A}=-\frac{5}{3}$ and $\mathrm{B}=-\frac{2}{3}$.

Secondly, putting $x^2=-4$, (N) becomes

$$-3(\mathrm{C}x+\mathrm{D})=-5x-11;$$

$$\therefore\ \mathrm{C}x+\mathrm{D}=\frac{5}{3}x+\frac{11}{3},\quad \therefore\ \mathrm{C}=\frac{5}{3},\text{ and }\mathrm{D}=\frac{11}{3}.$$

NOTE.—In using this method, it is unnecessary to substitute for x its imaginary value; we keep the symbol x, merely remembering that it *is* imaginary, but we substitute for x^2 wherever it occurs. The fact that x is imaginary makes the resulting equation an identity, so that we might write

$$\mathrm{A}x+\mathrm{B}\equiv-\tfrac{1}{3}(5x+2),\quad\text{and}\quad \mathrm{C}x+\mathrm{D}\equiv\tfrac{1}{3}(5x+11).$$

---

* It is not *necessary* to make the two sides exactly divisible: we may divide both sides of (N) as it stands by either x^2+4 or x^2+1 and note that the quotients on the two sides must be identically equal, and so likewise must the two remainders. Perhaps this is the neatest way of performing the operation.

The original fraction, viz.

$$\frac{3x^2-5x+1}{(x^2+1)(x^2+4)},$$

is therefore equal to

$$\frac{1}{3}\left\{\frac{5x+11}{x^2+4}-\frac{5x+2}{x^2+1}\right\}.$$

This is the simplest form to which it can be reduced.

143. The n^{th} differential coefficient of a fraction with a quadratic denominator is not simple, and as it is not practically of great importance we shall only give it at the end of the chapter, to be omitted on first reading if desired. The integral is not very simple, but it is important. It consists of two parts, and depends on the two following formulae which the student already knows, or can easily verify by differentiation.

(1) The integral of $\dfrac{2x}{x^2+a^2}$ is $\log(x^2+a^2)$.

(2) " " $\dfrac{1}{x^2+a^2}$ is $\dfrac{1}{a}\tan^{-1}\dfrac{x}{a}$.

Thus the integral of $\dfrac{5x}{x^2+4}$ is $\dfrac{5}{2}\log(x^2+4)$.

" $\dfrac{11}{x^2+4}$ is $\dfrac{11}{2}\tan^{-1}\dfrac{x}{2}$.

" $\dfrac{5x}{x^2+1}$ is $\dfrac{5}{2}\log(x^2+1)$.

" $\dfrac{2}{x^2+1}$ is $2\tan^{-1}x$.

∴ the integral of $\dfrac{3x^2-5x+1}{(x^2+1)(x^2+4)}$

is

$$\frac{5}{6}\log\frac{x^2+4}{x^2+1}+\frac{11}{6}\tan^{-1}\frac{x}{2}-\frac{2}{3}\tan^{-1}x.$$

In some of the following examples the integration is less simple, but this does not much matter, as it may well be left till the student is working at the integral calculus and has learnt how to treat all such fractions; the special object of this chapter is to show how to split up any given fractions into a number of simple fractions which shall be integrable by the proper methods. We shall, therefore, not attempt to integrate any of the remaining examples in the chapter.

144. Repeated quadratic factor in the denominator.

Such a fraction as $\dfrac{3x^3-5x^2+7x-9}{(x^2+x+1)^3}$

can be broken up into the sum of fractions of the form

$$\frac{A_1x+B_1}{x^2+x+1}+\frac{A_2x+B_2}{(x^2+x+1)^2}+\frac{A_3x+B_3}{(x^2+x+1)^3};$$

for if we add the three assumed fractions together, the numerator will be of the fifth degree, and the identification of this numerator with the given numerator will lead to just six equations among the six constants, from which they can be determined.

The equation of identity between the numerators is

$$(A_1x+B_1)(x^2+x+1)^2+(A_2x+B_2)(x^2+x+1)+(A_3x+B_3)$$
$$\equiv 3x^3-5x^2+7x-9. \qquad\ldots\ldots\ldots\ldots\ldots\ldots\text{(N)}$$

Using the method of equating coefficients we see at once that $A_1=0$ and $B_1=0$, since the coefficients of x^5 and x^4 are both zero.

Equating coefficients of x^3	$A_2=3.$
,, ,, ,, x^2	$A_2+B_2=-5,\ \therefore B_2=-8.$
,, ,, ,, x	$A_2+B_2+A_3=7,\ \therefore A_3=12.$
,, constant terms	$B_2+B_3=-9,\ \therefore B_3=-1.$

If the given numerator had contained higher powers of x, this method would have been troublesome. Probably the best method then would have been that of forced division, dividing both sides of (N) by x^2+x+1. This we will now do.

The remainder on the left is evidently A_3x+B_3, and that on the right will be found to be $12x-1$, the quotient being $3x-8$.

Hence $$A_3x+B_3\equiv 12x-1,$$
and $$(A_1x+B_1)(x^2+x+1)+(A_2x+B_2)\equiv 3x-8.$$

A repetition of the same method shows obviously that

$$A_2x + B_2 \equiv 3x - 8,$$

and $$A_1x + B_1 \equiv 0;$$

$\therefore$ the fraction may be replaced by

$$\frac{3x-8}{(x^2+x+1)^2} + \frac{12x-1}{(x^2+x+1)^3},$$

whose numerators are of the first degree only. The sum of these fractions is the simplest equivalent to the given fraction. The student will learn how to integrate such fractions when he comes to work systematically at the integral calculus.

NOTE.—The values of A_3 and B_3 above could have been found by the method of substitution, by putting

$$x^2 = -x - 1,$$

and, therefore, $$x^3 = -x^2 - x = 1;$$

whence $$A_3x + B_3 = 3 + 5(x+1) + 7x - 9$$
$$= 12x - 1.$$

This method is really a mere modification of the division method, not quite so good, usually, as actual division, for it gives the relation between the remainders only, instead of the relation between the remainders together with the relation between the quotients, both of which are simultaneously given by the division method. However, the method of substitution is interesting, and occasionally it may save trouble.

145. General Summary. The denominator of every rational fraction can be, in theory at any rate, broken up into first degree factors, or quadratic factors, or a mixture of the two, such factors being repeated or not. This we shall not prove, but take it as a known fact. Any proper fraction can, therefore, always be replaced by a series of partial fractions of the types already considered, and the numerators found by means of one or other of the preceding methods. The number of arbitrary constants which have to be assumed is in every case equal to the degree of the denominator of the given fraction.

An improper fraction can be replaced by a quotient + a series of simple fractions.

The general type of such equivalent quantities may be represented as under.

Let the given fraction be

$$\frac{f(x)}{(x-a)^r \dots \{(x-b)^2+c^2\}^s \dots}$$

Then it may be replaced by

$$\text{Quotient} + \left(\frac{A_1}{x-a} + \dots + \frac{A_r}{(x-a)^r}\right) + \dots$$
$$+ \left(\frac{B_1 x + C_1}{(x-b)^2+c^2} + \dots + \frac{B_s x + C_s}{\{(x-b)^2+c^2\}^s}\right) + \dots$$

The various methods of finding the numerators may be classified as

(1) The fundamental *method of equating coefficients* of the different powers of x in the equation of numerators which we have called (N).

(2) The method of substituting special values of x in the same equation.

(3) The method of forced division.

(4) Differentiation of both sides of the equation (N), followed by use of any of the other methods.

(5) The use of a formula in the case of an unrepeated linear factor. (See p. 150.)

EXAMPLES.

Replace the following fractions by their simplest equivalent partial fractions :

1. $\dfrac{3x^3+4x-2}{x^4+x^2+1}$.

2. $\dfrac{1}{x^4+1}$.

3. $\dfrac{3x^2+2}{x^4-x^2+1}$.

4. $\dfrac{x^4}{x^3+1}$.

5. $\dfrac{x^5}{x^3+27}$.

6. $\dfrac{x^2-7x-2}{(x-1)(x^2+1)^2}$.

7. $\dfrac{2x^2+3}{(x^3-1)^2}$.

8. $\dfrac{x^2-7x-2}{(x-1)(x^2+1)^3}$.

9. $\dfrac{3x^5-5x^3+7x-2}{(x^2+x+1)^3}$.

10. $\dfrac{x^4+9}{(x^2+4x+8)^3}$.

146. To find the n^{th} differential coefficient of $\dfrac{1}{x^2+1}$.

This can only be obtained in a manageable form by an artifice.

Let $x = \cot\theta$, and therefore $dx = -\operatorname{cosec}^2\theta\, d\theta$,

whence $$\mathrm{D}\theta = -\sin^2\theta,$$

if D is used for $\dfrac{d}{dx}$, for brevity.

Using this substitution, we have

$$y = \frac{1}{x^2+1} = \sin^2\theta;$$

$$\begin{aligned}\therefore\ \mathrm{D}y &= 2\sin\theta\cos\theta.\mathrm{D}\theta\\ &= -\sin^2\theta\sin 2\theta,\\ \mathrm{D}^2y &= -2(\sin\theta\cos\theta\sin 2\theta + \sin^2\theta\cos 2\theta)\mathrm{D}\theta\\ &= 2\sin^3\theta\sin 3\theta,\\ \mathrm{D}^3y &= 2\cdot3\sin^2\theta(\cos\theta\sin 3\theta + \sin\theta\cos 3\theta)\mathrm{D}\theta\\ &= -\underline{|3}\sin^4\theta\sin 4\theta,\end{aligned}$$

....

$$\mathrm{D}^ny = (-1)^n\underline{|n}\sin^{n+1}\theta\sin(n+1)\theta.$$

This is the formula.

The first integral of $\dfrac{1}{1+x^2}$, which we know is $\tan^{-1}x$, or, more generally, $\tan^{-1}x$ + any constant, can be deduced from the above formula by putting $n = -1$, and remembering that we may replace $\sin(n+1)\theta$ by $(n+1)\theta$ in this case. (Art. 32.)

We shall obtain $\mathrm{D}^{-1}y = -\theta$, the general integral being $-\theta$ + a constant, which agrees with the above, since

$$\tan^{-1}x = \frac{\pi}{2} - \theta.$$

147. To find the n^{th} differential coefficient of $\dfrac{x}{1+x^2}$.

This can be found by the same substitution, viz.

$$x = \cot\theta,\quad \mathrm{D}\theta = -\sin^2\theta;$$

whence, if $z = \dfrac{x}{1+x^2} = \dfrac{\cot\theta}{\operatorname{cosec}^2\theta} = \frac{1}{2}\sin 2\theta,$

$$\begin{aligned}
Dz &= -\sin^2\theta\cos 2\theta\,; \\
D^2z &= -2\sin\theta(\cos\theta\cos 2\theta - \sin\theta\sin 2\theta)D\theta \\
&= 2\sin^3\theta\cos 3\theta\,;
\end{aligned}$$

....

and, finally, $D^nz = (-1)^n \underline{|n}\sin^{n+1}\theta\cos(n+1)\theta.$

The above result might also have been deduced from the formula for $D^n\left(\dfrac{1}{1+x^2}\right)$ by help of Leibnitz' Theorem.

For, putting, as before, $y = \dfrac{1}{1+x^2}$, and $z = \dfrac{x}{1+x^2}$, we have

$$\begin{aligned}
z &= xy\,; \\
\therefore\ z_n &= xy_n + ny_{n-1} \\
&= (-1)^n \underline{|n}\sin^n\theta\{\cos\theta\sin(n+1)\theta - \sin n\theta\},
\end{aligned}$$

which is easily reduced to the previously obtained expression

$$(-1)^n \underline{|n}\sin^{n+1}\theta\cos(n+1)\theta,$$

since $\cos\theta\sin(n+1)\theta - \sin\theta\cos(n+1)\theta = \sin n\theta.$

The integral of $\dfrac{x}{1+x^2}$ is $\frac{1}{2}\log(1+x^2)$. This cannot be obtained readily by the artifice of putting $n = -1$ in the general expression above.

148. Various other expressions of the form $\dfrac{Ax+B}{(x^2+1)^r}$ can be expressed in terms of $\dfrac{1}{x^2+1}$ and $\dfrac{x}{x^2+1}$ and the successive differential coefficients of these functions, and thus expressions found for their n^{th} differential coefficients and first integrals. But other simpler methods of finding the first integrals of all such expressions will be given in a book on the integral calculus, so that the methods here given are rather interesting than important. Two examples are given below by way of illustration.

(1) *To find the n^{th} differential coefficient of* $\dfrac{1}{(x^2+1)^2}$.

By straightforward differentiation we find, after rearrangement,

$$D\frac{x}{x^2+1}=\frac{2}{(x^2+1)^2}-\frac{1}{x^2+1};$$

$$\therefore \frac{1}{(x^2+1)^2}=\frac{1}{2}\left(\frac{1}{x^2+1}+D\frac{x}{x^2+1}\right).$$

Hence $$D^n\frac{1}{(x^2+1)^2}=\frac{1}{2}\left(D^n\frac{1}{x^2+1}+D^{n+1}\frac{x}{x^2+1}\right)$$

and, in particular,

$$D^{-1}\frac{1}{(x^2+1)^2}=\frac{1}{2}\left(\tan^{-1}x+\frac{x}{x^2+1}\right).$$

(2) *To find the n^{th} differential coefficient of* $\dfrac{x}{(x^2+1)^3}$.

We have at once, by differentiation,

$$D\frac{1}{(x^2+1)^2}=-\frac{4x}{(x^2+1)^3};$$

$$\therefore D^n\frac{x}{(x^2+1)^3}=-\tfrac{1}{4}D^{n+1}\frac{1}{(x^2+1)^2}.$$

EXAMPLES.

Find the n^{th} differential coefficient of

1. $\dfrac{1}{x^2+a^2}$. 2. $\dfrac{x}{x^2+a^2}$. 3. $\dfrac{1}{(x-a)^2+b^2}$.

4. $\dfrac{x}{(x-a)^2+b^2}$. 5. $\dfrac{3x+4}{x^2+4x+8}$.

6. $\dfrac{1}{(x^2+4x+8)^2}$. 7. $\dfrac{x+2}{(x^2+4x+8)^2}$.

8. By differentiating $\dfrac{x}{(x^2+1)^2}$, find a formula for the n^{th} differential coefficient of $\dfrac{1}{(x^2+1)^3}$, and find its integral.

149. The following list of most of the more important results of this and the preceding chapter may be of use as a summary, and be of help to the student in the process of revision :

RECAPITULATION.

u	$D^{-1}u$	u	Du	$D^r u$
$\frac{}{\underline{r}}x^{n+r}$	$\frac{x^{n+1}}{n+1}$	x^n	nx^{n-1}	$\frac{\underline{\lvert n}}{\underline{\lvert n-r}}x^{n-r}$
$^{-1}+Bx^{r-2}$ $+\ldots+K$	constant	0	0	0
$\frac{r-1}{-1}\,\frac{(-1)^r}{x^{n-r}}$	$-\frac{1}{(n-1)x^{n-1}}$	$\frac{1}{x^n}$	$-\frac{n}{x^{n+1}}$	$\frac{\underline{\lvert n+r-1}}{\underline{\lvert n-1}}\,\frac{(-1)^r}{x^{n+r}}$
$\log x-1-\frac{1}{2}-\ldots-\frac{1}{r}\Big]$	$x(\log x-1)$	$\log x$	$\frac{1}{x}$	$\frac{\underline{\lvert r-1}\,(-1)^{r-1}}{x^r}$
$\left(x-\frac{r\pi}{2}\right)$	$\sin\left(x-\frac{\pi}{2}\right)$ $=-\cos x$	$\sin x$	$\sin\left(x+\frac{\pi}{2}\right)$ $=\cos x$	$\sin\left(x+\frac{r\pi}{2}\right)$
in$\left(mx-\frac{r\pi}{2}\right)$	$\frac{1}{m}\sin\left(mx-\frac{\pi}{2}\right)$ $=-\frac{\cos mx}{m}$	$\sin mx$	$m\sin\left(mx+\frac{\pi}{2}\right)$ $=m\cos mx$	$m^r\sin\left(mx+\frac{r\pi}{2}\right)$
os$\left(mx-\frac{r\pi}{2}\right)$	$\frac{1}{m}\cos\left(mx-\frac{\pi}{2}\right)$ $=\frac{\sin mx}{m}$	$\cos mx$	$m\cos\left(mx+\frac{\pi}{2}\right)$ $=-m\sin mx$	$m^r\cos\left(mx+\frac{r\pi}{2}\right)$
^{ax}	$\frac{1}{a}e^{ax}$	e^{ax}	ae^{ax}	$a^r e^{ax}$
$^x\sin(bx-r\alpha)$	$\frac{1}{k}e^{ax}\sin(bx-\alpha)$	$e^{ax}\sin bx$	$ke^{ax}\sin(bx+\alpha)$ where $k=\surd(a^2+b^2)$ and $\tan\alpha=b\div a$	$k^r e^{ax}\sin(bx+r\alpha)$
	$\frac{1}{a}\log(ax+b)$	$\frac{1}{ax+b}$	$-\frac{a}{(ax+b)^2}$	$\frac{(-a)^r\,\underline{\lvert r}}{(ax+b)^{r+1}}$
	$\tan^{-1}x$ or $-\cot^{-1}x$	$\frac{1}{x^2+1}$	$\frac{-2x}{(x^2+1)^2}$	$(-1)^r\,\underline{\lvert r}\sin^{r+1}\theta\sin(r+1)\theta$ where $\cot\theta=x$
	$\frac{1}{2}\log(x^2+1)$	$\frac{x}{x^2+1}$	$\frac{1-x^2}{(x^2+1)^2}$	$(-1)^r\,\underline{\lvert r}\sin^{r+1}\theta\cos(r+1)\theta$
$+D_2)^{-r}uv$	$(D_1+D_2)^{-1}uv$	uv	$(D_1+D_2)uv$ $=Du.v+u.Dv$	$(D_1+D_2)^r uv$
$+D)^{-r}u$	$e^{ax}(a+D)^{-1}u$	$e^{ax}u$	$e^{ax}(a+D)u$	$e^{ax}(a+D)^r u$

CHAPTER XI.

TAYLOR'S AND MACLAURIN'S THEOREMS.

150. Taylor's Theorem. We know that when h is small, the formula

$$f(a+h)-f(a)=hf'(a)$$

is approximately true, being true to the first power of h.

[See Chap. VI., also Chap. I., in which dx is used instead of h, and powers of dx beyond the first are neglected.]

We desire now to find the complete expression for

$$f(a+h)-f(a),$$

so as to be able to use it for finite values of h. This complete expression will have terms containing h^2, h^3, etc.

Thus $$(a+h)^2-a^2=h(2a)+h^2;$$

and $$(a+h)^3-a^3=h(3a^2)+h^2(3a)+h^3.$$

We will establish a formula which will enable us to determine the coefficients of the different powers of h in every case where such a series is available, *i.e.* in the case of all functions which obey certain conditions.

The formula, which is called Taylor's Theorem for the expansion of a function of $a+h$ in powers of h, is

$$\mathbf{f(a+h)=f(a)+hf'(a)+\frac{h^2}{1.2}f''(a)+\frac{h^3}{1.2.3}f'''(a)+\ldots.}$$

The student should verify that this formula holds in the case of the above two examples, and, before proving it, he might make himself familiar with it by working the examples below.

EXAMPLES.

1. Expand $(a+h)^n$ in powers of h by Taylor's Theorem. [It will be found that this gives us the Binomial Theorem, which is thus seen to be a particular case of Taylor's Theorem.]

2. If $x=a+h$, arrange $x^4-3x^3+7x^2$ in powers of h by actual expansion, and show that the terms follow the law given by Taylor's Theorem.

3. Expand e^{a+h} in powers of h, and compare the result with the expansion of e^h given in Chap. IV.

4. Expand $\sin(\alpha+\beta)$ in powers of β, and hence deduce the expansions of $\sin\beta$ and $\cos\beta$ in powers of β.

5. Expand $\log_e(a+h)$ in powers of h, and deduce the expansion of $\log_e(1+x)$.

6. Expand $\log\sin(x+h)$ in powers of h as far as the term involving h^3 inclusive.

151. Mode of obtaining Taylor's Theorem. There are several instructive ways of obtaining Taylor's Theorem. The simplest way is to use the method we have before adopted to find the expansions of e^x, $\sin x$, and other simple functions, *i.e.* to assume an expansion in powers of h with unknown coefficients, and then proceed to find what those coefficients must be. This method presupposes that such expansion is possible, which, we shall see later, is not always the case.

Assume $f(a+h)-f(a)=\mathrm{A}_1h+\mathrm{A}_2h^2+\mathrm{A}_3h^3+\dots\dots\dots\dots$(1)

This, by supposition, is an identity, and therefore if we differentiate both sides the resulting expression will also be an identity.

Differentiate both sides with regard to h, keeping a constant.

$$\therefore\ f'(a+h)\,dh=(\mathrm{A}_1+2\mathrm{A}_2h+3\mathrm{A}_3h^2+\dots)\,dh;$$

$$\therefore\ f'(a+h)=\mathrm{A}_1+2\mathrm{A}_2h+3\mathrm{A}_3h^2+4\mathrm{A}_4h^3+\dots\dots\dots \qquad (2)$$

This being an identity, we are again entitled to differentiate both sides with regard to h, obtaining another identity; and so on as many times as we please. We thus find

$$f''(a+h)=1.2\mathrm{A}_2+2.3\mathrm{A}_3h+3.4\mathrm{A}_4h^2+\dots;\ \dots\dots\dots\dots \qquad (3)$$

$$f'''(a+h)=1.2.3\mathrm{A}_3+2.3.4\mathrm{A}_4h+3.4.5\mathrm{A}_5h^2+\dots;\ \dots\dots \qquad (4)$$

$$f^{iv}(a+h)=1.2.3.4\mathrm{A}_4+2.3.4.5\mathrm{A}_5h+\dots;\ \dots\dots\dots\dots\dots \qquad (5)$$

Now, as these are, by supposition, identities, they must hold for all values of h. Hence, putting $h=0$ in (2), (3), (4), ..., we find

$$f'(a)=A_1;$$
$$f''(a)=1.2A_2;$$
$$f'''(a)=1.2.3A_3;$$
$$f^{iv}(a)=1.2.3.4A_4;$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

These results give us the values of the successive coefficients; and inserting these in (1) we arrive at the required formula, viz.:

$$f(a+h)-f(a)=hf'(a)+\frac{h^2}{1.2}f''(a)+\frac{h^3}{1.2.3}f'''(a)+\dots.$$

NOTE.—In the above work we have assumed that all the differential coefficients of $f(a+h)$ are functions which are finite when $h=0$ and are continuous throughout the range of values of h, as if they are not continuous we cannot differentiate them. If they are all finite and continuous throughout the range, and if, in addition, the terms of the series which contain high powers of h become unimportant, the theorem is both true and numerically useful.

152. Example illustrating the limitations of Taylor's Theorem. It is easy to give an illustration of an expansion which is true and intelligible for certain values of h, but not for others. Thus

$$\frac{1}{1-h}=1+h+h^2+h^3+\dots,$$

as can be proved by means of the theorem, or, more simply, by actual division.

Now, this apparent identity is true when h is less than unity, but if h is greater than unity we get a negative quantity on the left-hand side and an infinitely great positive quantity on the right-hand side, so that the two sides cannot in this case be equal. It is noteworthy, in connection with the failure of the series when h is greater than 1, that, when $h=1$, both $\frac{1}{1-h}$ and its differential coefficients become infinite and some of them discontinuous, having great positive values when h is just less than 1, and great negative values when h is just greater than 1.

This discrepancy between the two sides is very curious in such a simple example. It shows how very careful we must be. We can find out how it arises in the present case, and will be able to apply a correction which will restore the identity of the two sides of the equation.

For, if we go through the process of division, we find

$$\begin{aligned}\frac{1}{1-h} &= 1+\left(\frac{h}{1-h}\right)\\ &= 1+h+\left(\frac{h^2}{1-h}\right)\\ &= \dots\dots\dots\dots\dots\\ &= 1+h+h^2+h^3+\dots+h^{n-1}+\left(\frac{h^n}{1-h}\right).\end{aligned}$$

Here we have an identity whatever h may be. Moreover we have, not an infinite series, but as many terms of the series as we please, followed by a remainder which is not in the form of a series.

Now, when h is less than 1, the remainder after n terms of the series can be made as unimportant as we please, by increasing n.

But if h is greater than 1, the remainder becomes more and more important the greater n is, and is moreover negative. The terms of the series together with the remainder are equal to the given function, although the sum of the terms by themselves is more and more remote from the value of the function the greater we take n.

We are led to the conclusion that the infinite series represents the function properly when the remainder after n terms can be made as small as we please by increasing n, but not otherwise. The conclusion is consequently inevitable that the function $\frac{1}{1-x}$ and the series $1+x+x^2+\dots$ to ∞ are not *identical*, but are equal to each other through a certain range, and entirely unequal beyond that range.

In fact the real identity is

$$\frac{1}{1-x}-\frac{x^n}{1-x}=1+x+x^2+\ldots+x^{n-1},$$

being true for all values of n and all values of x whether great or small.

If we divide this identity by x^n we have the identity

$$\frac{1}{x^n(1-x)}-\frac{1}{1-x}=\frac{1}{x}+\frac{1}{x^2}+\ldots+\frac{1}{x^n}.$$

From the first of these two identities we see that, when x is numerically less than 1,

$$\frac{1}{1-x}=1+x+x^2+\ldots \text{ to infinity},$$

and from the second we see that, when x is numerically greater than 1,

$$\frac{1}{1-x}=-\left(\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3}+\ldots \text{ to infinity}\right),$$

showing that, when an infinite series of positive powers of x fails to represent a function, it may be possible to expand it in a series of negative powers. In this case the expansion in negative powers could have been obtained by putting $z=\frac{1}{x}$ in the function, and then expanding in positive powers of z.

The student should draw the graph of $\frac{1}{1-x}$, and also of $1+x$, $1+x+x^2$, $1+x+x^2+x^3$, etc.

He will see that by taking more and more terms of the series the corresponding graph lies closer and closer to the graph of $\frac{1}{1-x}$ between the limits $x=\pm 1$, but not beyond those limits.

He should notice also that when $x=-1$, the ordinate of the function graph has the value $\frac{1}{2}$, whereas those of the series graphs have alternately the values 1 and 0.

In the limit when we take the graph of the infinite series, it will coincide with the graph of $\frac{1}{1-x}$ between the values

$x = \pm 1$, but is infinite beyond those limits; and, when $x = -1$, its ordinate is not $\frac{1}{2}$ but either 1 or 0 according to whether we suppose the last term to be an even or an odd power of x.

The series therefore is equal to the function throughout the range for which the distant terms of the series are unimportant.

The student might also draw the curves

$$y = -\frac{1}{x}, \quad y = -\frac{1}{x} - \frac{1}{x^2}, \text{ etc.},$$

from which he will see that the more terms he takes the nearer will the graph approach the graph of the function *beyond* the limits $x = \pm 1$.

This example has been dwelt on at some length to show clearly that Taylor's infinite series is *not always* identically equal to the function from which it is derived, *but only for a certain range of values of* x, and to show that the apparently simple proof given in Art. 151 slurs over a good number of difficulties and limitations.

153. Second proof of Taylor's Theorem. Another instructive mode of arriving at Taylor's series, which will indicate the need that $f(x)$ and all its differential coefficients must be continuous and finite throughout the range of values of x contained between a and $a+h$ inclusive, and that the end of the series must be unimportant, is by gradually building up the finite increment h by a succession of infinitely small equal increments dx. We will first take these increments small but not infinitely so, calling them δx, and then find what the series becomes in the limit when they are infinitely small and infinite in number, their sum being equal to h.

Denoting $f(a)$ by y for brevity, and thinking of the values of the function as ordinates of the graph of $f(x)$, the ordinate corresponding to $x = a + \delta x$ will be $y + \delta y$, and the next ordinate, viz. when $x = a + 2\delta x$, will be

$$y + \delta y + \delta(y + \delta y) = y + 2\delta y + \delta(\delta y),$$

which is written $y + 2\delta y + \delta^2 y$ (see Chap. VIII., p. 121).

Similarly, when $\quad x = a + 3\delta x$,

the ordinate $\quad = y + 2\delta y + \delta^2 y + \delta(y + 2\delta y + \delta^2 y)$

$\quad = y + 3\delta y + 3\delta^2 y + \delta^3 y$, and, when $x=a+m\delta x$, the ordinate

$$=y+m\delta y+\frac{m(m-1)}{1.2}\delta^2 y+\frac{m(m-1)(m-2)}{1.2.3}\delta^3 y+\dots,$$

the coefficients being evidently the same as those which occur in the Binomial Theorem.

If, now, we make $m\delta x=h$, we obtain

$$f(a+h)=y+h\frac{\delta y}{\delta x}+\frac{h^2\left(1-\frac{1}{m}\right)}{1.2}\frac{\delta^2 y}{(\delta x)^2}$$
$$+\frac{h^3\left(1-\frac{1}{m}\right)\left(1-\frac{2}{m}\right)}{1.2.3}\frac{\delta^3 y}{(\delta x)^3}+\dots$$

Now let δx be made infinitely small, and consequently m infinitely great: the series becomes

$$f(a+h)=y+h\frac{dy}{dx}+\frac{h^2}{1.2}\frac{d^2y}{dx^2}+\dots$$
$$=f(a)+hf'(a)+\frac{h^2}{1.2}f''(a)+\dots$$

We here assume that neither the function nor any of its differential coefficients become infinite or discontinuous between $x=a$ and $x=a+h$, inclusive, and that the infinitely distant part of the series is unimportant.

If these conditions are satisfied, the series is equal to $f(a+h)$.

NOTE.—It is not essential that the *limiting value* of $f^n(x)$ shall be finite when $n=\infty$, so long as the limiting value of $h^n f^n(x)\div \underline{|n}$ is zero. But, of course, $f^n(x)$ must not be infinite for any *finite* value of n.

In many cases the conditions will be satisfied for a certain range of values of a and h, but not for all values. So, to repeat the previous caution, we cannot say that the series *is* the function, but only that through a certain range the two are equal in value.

154. In some cases we may not be able to form all the differential coefficients of a function, so that we may not be able to tell, except indirectly, whether all of them are finite within the range of values required.

But if the function and a certain number (n) of its differential coefficients do fulfil the condition, we can find an approximate expression for the remainder of the series which will enable us to estimate to what extent the sum of the first n terms give an approximate value of $f(a+h)$. We shall find, by help of this approximate remainder, that, in practically all cases, when h is small a few terms of the series will be a real approximation to the value of $f(a+h)$.

This discovery is of great importance, as one of the chief uses made of the series is to find, when h is small, but not small enough for us to be content with its first power only, by how much $f(a+h)$ differs from $f(a)$. It is therefore very important to know to some extent the limits of the approximation obtained when we use only a few terms of the infinite series.

155. Taylor's Theorem with the Remainder after *n* terms.

Suppose $f(x)$ and its first n differential coefficients are finite and continuous between $x=a$ and $x=a+h$. We shall prove that

$$\mathbf{f(a+h)=f(a)+hf'(a)+\frac{h^2}{1.2}f''(a)+\dots}$$

$$\mathbf{+\frac{h^{n-1}}{\underline{|n-1}}f^{n-1}(a)+\frac{h^n}{\underline{|n}}f^n(a+\theta h),}$$

where θ is a positive proper fraction, so that $a+\theta h$ is intermediate in value between a and $a+h$.

Assume

$$f(a+h)=f(a)+hf'(a)+\dots+\frac{h^{n-1}}{\underline{|n-1}}f^{n-1}(a)+\frac{h^n}{\underline{|n}}\mathrm{H}. \quad \dots\dots(1)$$

We want to prove that $\mathrm{H}=f^n(a+\theta h)$.

Take a new variable z, whose values are to be considered between $z=0$ and $z=h$, and let

$$\phi(z)=f(a+z)+(h-z)f'(a+z)+\dots$$

$$+\frac{(h-z)^{n-1}}{\underline{|n-1}}f^{n-1}(a+z)+\frac{(h-z)^n}{\underline{|n}}\mathrm{H}. \quad \dots\dots(2)$$

This is obtained from the right-hand side of (1) by putting $a+z$ for a, and $h-z$ for h, and consequently $\phi(z)=f(a+h)$ both when $z=0$ and when $z=h$, but assumes different values between these limits.

Therefore for some intermediate value of z, say $z=\theta h$, when θ denotes a positive proper fraction, $\phi(z)$ must have a maximum or minimum value, and $\phi'(\theta h)$ will be zero, since $\phi(z)$, and consequently $\phi'(z)$, are finite and continuous throughout the range.

Now $$\phi'(z)=\frac{(h-z)^{n-1}}{\underline{|n-1}}\{f^n(a+z)-\mathrm{H}\},$$

since all its other terms cancel in pairs.

$$\therefore\ \mathrm{H}=f^n(a+\theta h),\ \text{which was to be proved.}$$

[This method of proof was given by Prof. Harding in the *Mathematical Gazette*, Oct. 1908, pp. 332, 3.]

EXAMPLES.

1. Assuming, as on p. 55, that $De^x=e^x$, prove that

$$e^{a+h}=e^a\left(1+h+\frac{h^2}{\underline{|2}}+\ldots+\frac{h^{n-1}}{\underline{|n-1}}+\frac{h^n}{\underline{|n}}e^{\theta h}\right),$$

and write down the corresponding series for e^x, by putting $a=0$, and $h=x$.

2. Prove that

$$\sin(a+h)=\sin a+h\sin\left(a+\frac{\pi}{2}\right)+\frac{h^2}{\underline{|2}}\sin\left(a+\frac{2\pi}{2}\right)+\ldots$$
$$+\frac{h^{n-1}}{\underline{|n-1}}\sin\left(a+\frac{(n+1)\pi}{2}\right)+\frac{h^n}{\underline{|n}}\sin\left(a+\theta h+\frac{n\pi}{2}\right),$$

and deduce, by putting $a=0$, and $h=x$, that

$$\sin x=x-\frac{x^3}{\underline{|3}}+\frac{x^5}{\underline{|5}}-\ldots+\frac{x^n}{\underline{|n}}\sin\left(\theta x+\frac{n\pi}{2}\right).$$

3. Obtain similar series for $\cos(a+h)$ and $\cos x$.

It is to be observed that the form of Taylor's series thus proved is not an infinite series at all, but a finite number of terms (of the same form as those of the infinite series) followed by a remainder.

This remainder is not a perfectly definite expression, like that which was found in connection with the function of Art. 152, as we only know that θ is *some* proper fraction, but nevertheless it is an *indication* of the error to which we are liable if we stop short at n terms of the series. The error cannot be greater than $\frac{h^n}{\underline{|n}}$ multiplied by the greatest numerical value of $f^n(x)$ between the limits $x=a$ and $x=a+h$, and it cannot be less than $\frac{h^n}{\underline{|n}}$ multiplied by the least numerical value of $f^n(x)$ between those limits.

It must not be overlooked that the above proof requires that $f(x)$ and all its derivatives as far as $f^n(x)$ shall be finite and continuous throughout the range $x=a$ to $x=a+h$, since $\phi(z)$ and its derivative $\phi'(z)$ involve all these quantities.

[NOTE.—It would have been sufficient, however, to assert that $f^n(x)$ must be finite and continuous throughout the range, for a little consideration will show that if any derivative became discontinuous or infinite, all the succeeding ones must follow suit. Hence, if $f^n(x)$ satisfies the conditions, all its predecessors must also satisfy the conditions.]

156. If *all* the derivatives of $f(x)$ are finite and continuous for all finite values of n, however great, and if, also, by increasing n we can make $\frac{h^n}{\underline{|n}}f^n(a+\theta h)$ as small as we please (even though $f^n(a+\theta h)$ may itself *tend* to be infinite), we may proceed to the infinite series for $f(a+h)$.

[*N.B.*—If, when $x=\infty$, the limiting value of $f^n(a+\theta h)$ is itself finite, the above condition is *à fortiori* satisfied. This is the case with the functions e^x, $\sin x$, and $\cos x$, for all values of x.]

Hence, to recapitulate:

If $f(x)$ and all its derivatives to the n^{th} are finite and continuous between $x=a$ and $x=a+h$,

$$f(a+h)=f(a)+hf'(a)+\ldots+\frac{h^n}{\underline{|n}}f^n(a+\theta h);$$

and if, further, by increasing n, we can make $\frac{h^n}{\underline{|n}}f^n(a+\theta h)$ as small as we please, we shall have

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{1.2}f''(a)+\ldots \text{ to infinity.}$$

The two series are not identical.

One is called Taylor's series with a remainder after n terms, and the other, Taylor's series to infinity.

157. Convergency of the series. When, on increasing n, we can make $\frac{h^n}{\underline{|n}}f^n(a+\theta h)$ as small as we please, the infinite series

$$f(a)+hf'(a)+\frac{h^2}{1.2}f''(a)+\ldots \text{ to infinity}$$

is said to be convergent. Its sum is finite and equal to $f(a+h)$.

If the condition is satisfied for all values of a and h, the series is said to be universally convergent; if only for a particular range of values, it is said to be conditionally convergent. Thus the series for $\frac{1}{1-x}$ is conditionally convergent; the series for e^x, $\sin x$, and $\cos x$ are all universally convergent.

158. Use of Taylor's Theorem for Tabulation. Illustrations of the use of Taylor's Theorem will be given in next chapter. We will however give some examples in this and the two following articles.

Suppose it required to form a table of values of $x(144-x^2)$ for values of $x=1{\cdot}5$, $1{\cdot}6\ldots$to $2{\cdot}5$. We may work as follows:

$$\begin{array}{lrl}
\text{Let} & f(x)=x(144-x^2); & \therefore f(2)=280; \\
 & \therefore f'(x)=144-3x^2; & \therefore f'(2)=132; \\
 & f''(x)=-6x; & \therefore f''(2)=-12; \\
 & f'''(x)=-6; & \therefore f'''(2)=-6.
\end{array}$$

Now, by Taylor's Theorem,

$$f(2+h)=f(2)+hf'(2)+\frac{h^2}{1.2}f''(2)+\ldots;$$

$$\therefore\ f(2+h)=280+132h-6h^2-h^3;$$

$$\therefore\ f(2{\cdot}1)=280+13{\cdot}2-0{\cdot}06-0{\cdot}001=293{\cdot}139,$$
$$f(2{\cdot}2)=280+26{\cdot}4-0{\cdot}24-0{\cdot}008=306{\cdot}152,$$
$$f(2{\cdot}3)=280+39{\cdot}6-0{\cdot}54-0{\cdot}027=319{\cdot}033,$$

and so on.

Also
$$f(1{\cdot}9)=280-13{\cdot}2-0{\cdot}06+0{\cdot}001=266{\cdot}741,$$
$$f(1{\cdot}8)=280-26{\cdot}4-0{\cdot}24+0{\cdot}008=253{\cdot}368,$$
$$f(1{\cdot}7)=280-39{\cdot}6-0{\cdot}54+0{\cdot}027=239{\cdot}887.$$

……………………………………

It is obvious that these calculations are much simpler than the disconnected work which would be required by straightforward evaluation without the help of the series.

159. Use of Taylor's Theorem for approximating to roots of equations. As another example, let us find an approximation to the root of an equation $f(x)=0$ when we know a first approximation $x=a$.

Suppose $f(a)=\text{A}$, where A is small, and that $f(a+h)=0$, where h is a small correction to be found.

The equation is $f(a)+hf'(a)+\frac{h^2}{\underline{|2}}f''(a)+\ldots=0.$

or say, for brevity $\text{A}+\text{B}h+\text{C}h^2+\ldots=0,$

and let us not go beyond the square of h.

The first approximation, neglecting h^2, is $h=-\frac{\text{A}}{\text{B}}$.

This is the approximation given in Chap. VI.

An improvement can be found by writing the equation for h in the form

$$\text{B}h=-\text{A}-\text{C}h^2;$$

$$\therefore\ h=-\frac{\text{A}}{\text{B}}-\frac{\text{C}h^2}{\text{B}}.$$

Now, on the right-hand side insert for h its first approximate value, $-\mathrm{A} \div \mathrm{B}$, and we obtain as a closer approximation,

$$h = -\frac{\mathrm{A}}{\mathrm{B}} - \frac{\mathrm{CA}^2}{\mathrm{B}^3},$$

i.e.
$$h = -\frac{\mathrm{A}}{f'(a)} - \frac{\mathrm{A}^2}{1.2} \frac{f''(a)}{(f'(a))^3}.$$

In this way h could be expressed in a series of powers of A. The above is correct as far as A^2. The next term would involve A^3, but is not simple enough to be of much use.

The root is
$$a - \frac{\mathrm{A}}{f'(a)} - \frac{\mathrm{A}^2}{1.2} \frac{f''(a)}{(f'(a))^3},$$
where $\mathrm{A} = f(a)$.

Probably in practice it would be better to express the root as

$$a + h_1 - \frac{\frac{1}{2} f''(a)}{f'(a)} h_1^2, \text{ where } h_1 = -\frac{f(a)}{f'(a)}.$$

Example. Find by the above formula the roots of the equation $x^3 - 4x^2 - 11x + 32 = 0$, being given that the approximate roots are 5, 2, and -3.

Ans. 4·913, 2·137, and -3·0492.

160. Taylor's Theorem in connection with the theory of maxima and minima. In Arts. 106, 107, a method of determining the conditions that $f(a)$ should be a maximum or a minimum value of $f(x)$, depending on the successive derivatives $f'(a)$, $f''(a)$,... was partially indicated. We are now in a position to state the whole theory of the method.

If $f(a)$ is a maximum value of $f(x)$, it must be greater than all neighbouring values, *i.e.* greater than both $f(a+h)$ and $f(a-h)$, for small values of h, and similarly if $f(a)$ is a minimum, it must be less than both $f(a+h)$ and $f(a-h)$.

Now
$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{\underline{|2}} f''(a) + \ldots,$$

and
$$f(a-h) = f(a) - hf'(a) + \frac{h^2}{\underline{|2}} f''(a) - \ldots.$$

Hence when h is small we have, approximately,

$$f(a+h)=f(a)+hf'(a),$$

and

$$f(a-h)=f(a)-hf'(a).$$

One of these will be greater than $f(a)$ and the other less unless $f'(a)=0$.

Therefore, the first condition that $f(a)$ shall be a maximum or minimum value of $f(x)$ is that $f'(a)=0$.

In that case $f(a+h)=f(a)+\frac{h^2}{\underline{|2}}f''(a)$, approximately,

and $f(a-h)=f(a)+\frac{h^2}{\underline{|2}}f''(a)$, approximately.

Hence, if $f''(a)$ is positive, $f(a)$ is a minimum, being obviously less than both $f(a+h)$ and $f(a-h)$.

If, on the contrary, $f''(a)$ is negative, $f(a)$ is a maximum.

But if $f'(a)=0$ and also $f''(a)=0$, then

$$f(a+h)=f(a)+\frac{h^3}{\underline{|3}}f'''(a), \text{ approximately,}$$

and

$$f(a-h)=f(a)-\frac{h^3}{\underline{|3}}f'''(a), \text{ approximately.}$$

Hence in this case $f(a)$ cannot be a maximum or minimum value of $f(x)$ unless $f'''(a)=0$.

Proceeding in this way we see that $f(a)$ cannot be a maximum or minimum unless the first derivative which does not vanish is of even order, say $f^{2n}(a)$.

And *then* we have both

$$f(a+h) \text{ and } f(a-h)=f(a)+\frac{h^{2n}}{\underline{|2n}}f^{2n}(a), \text{ approximately,}$$

and therefore $f(a)$ will be a minimum if $f^{2n}(a)$ is positive,

and $f(a)$ will be a maximum if $f^{2n}(a)$ is negative.

This is the theory of the investigation by means of the uccessive derivatives.

161. Special case of Taylor's Theorem. Geometrical illu tration. If, in Taylor's Theorem with remainder, we put $n=$ we obtain the equation

$$\mathbf{f(a+h)-f(a)=hf'(a+\theta h).}$$

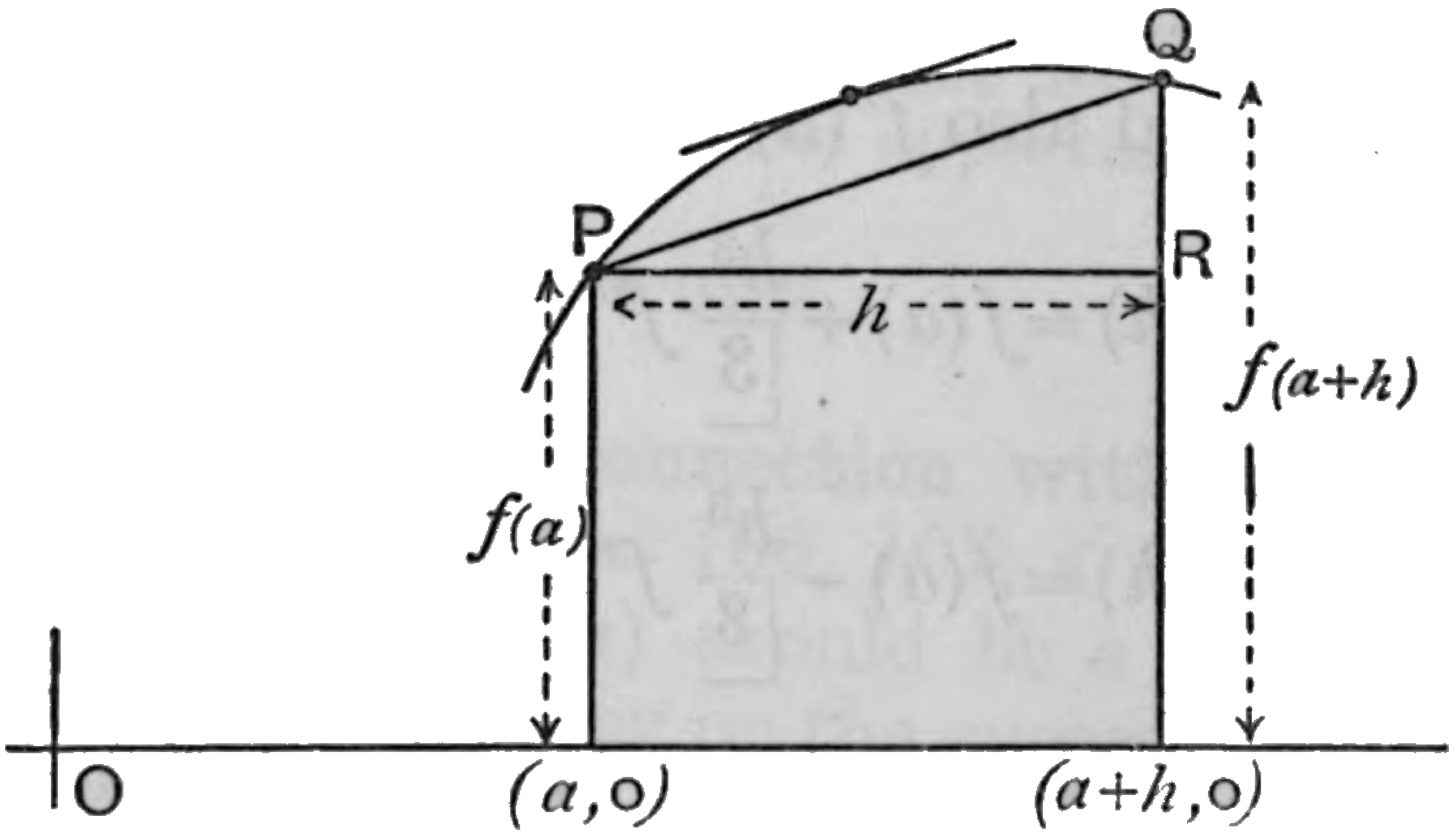
This is an interesting equation as it shows that tl approximate relation for small values of h, viz.

$$f(a+h)-f(a)=hf'(a),$$

might be improved by taking on the right, not $f'(a)$, but som value of the derivative $f'(x)$ *between* $x=a$ and $x=a+h$.

This fact can be illustrated geometrically, in such way as t indicate, in the case of a function whose graph is drawn, fc what value of x, approximately, the derivative should be chose

If we draw the curve $y=f(x)$, and take on it two point P and Q whose abscissae are respectively a and $a+h$, th ordinates of P and Q will be $f(a)$ and $f(a+h)$, respectively.



Consequently the gradient of the chord PQ will be

$$\frac{f(a+h)-f(a)}{h}.$$

Now, evidently, if the gradients of the curve between P ar Q are finite and vary continuously, *i.e.* with no break in tl back of the curve, the tangent to the curve at some poi between P and Q must be parallel to the chord PQ, *i.e.* tl gradient of the curve at this intermediate point will be equ to the gradient of PQ.

But $f'(a+\theta h)$, where θ is a proper fraction, just denotes gradient of the curve at some such intermediate point. He the equation with which this article started merely asserts the above geometrical fact, viz. that between P and Q there must be some point on the curve where the tangent is parallel to PQ.

162. Maclaurin's theorem. If in Taylor's theorem we put $a=0$, and write x for h, we obtain the series

$$\mathbf{f(x)=f(0)+xf'(0)+\frac{x^2}{1.2}f''(0)+\ldots}$$
$$\mathbf{+\frac{x^{n-1}}{\underline{|n-1}}f^{n-1}(0)+\frac{x^n}{\underline{|n}}f^n(\theta x),}$$

when $f(x)$ and all its derivatives to the n^{th}, inclusive, are finite and continuous between 0 and x.

Also

$$\mathbf{f(x)=f(0)+xf'(0)+\frac{x^2}{1.2}f''(0)+\ldots\text{ to infinity,}}$$

if, on increasing n, the remainder $\dfrac{x^n}{\underline{|n}}f^n(\theta x)$ can be made as small as we please.

This theorem, in its two forms, is called Maclaurin's Theorem, which is thus seen to be a special case of Taylor's Theorem. It enables us to express any function of x, which obeys the necessary conditions, in powers of x.

The expansions of e^x, $\sin x$, $\cos x$, ... are illustrations of the theorem.

A useful, compact, way of stating the theorem is as follows: If y is a function of x, and y_1, y_2, ... are its successive derivatives, and, if a_0, a_1, a_2, ... denote the values of y, y_1, y_2, ... when $x=0$, then

$$y=a_0+a1x+a_2\frac{x^2}{1.2}+\ldots\text{ to infinity,}$$

provided none of the terms are infinite and the series is onvergent.

Example. To find the expansion of $\sin x$.

Let $y=\sin x;\ \therefore\ a_0=0;$

$\therefore\ y_1=\cos x,\quad a_1=1,$

$y_2=-\sin x,\quad a_2=0,$

$y_3=-\cos x,\quad a_3=-1.$

Continuing, it will be found that y_4, y_5, y_6, y_7 are a repetition of y, y_1, y_2, and y_3, in the same order, and so on.

Hence
$$\begin{aligned} a_0 &= a_4 = a_8 = \ldots = 0, \\ a_1 &= a_5 = a_9 = \ldots = 1, \\ a_2 &= a_6 = a_{10} = \ldots = 0, \\ a_3 &= a_7 = a_{11} = \ldots = -1; \end{aligned}$$

∴ finally,
$$y = x - \frac{x^3}{\underline{\lvert 3}} + \frac{x^5}{\underline{\lvert 5}} - \frac{x^7}{\underline{\lvert 7}} + \ldots.$$

This series holds for all values of x, since $\sin x$ and its derivatives are always finite. Any function whose successive derivatives can be found can be expanded in the same way, provided neither the function nor any of its derivatives become infinite when $x=0$. The series so found will be a proper representation of the function for such range of values of x, from $x=0$ onwards, as include no infinite values of the function or its derivatives.

The student should form the series for $\log_e(1+x)$ in the same way, and note that the series will only properly represent this function between $x=\pm 1$.

As a harder example we may take the following:

To expand $\tan x$ *in powers of* x, *up to* x^7 *inclusive.*

One way of forming the successive derivatives in this case has been given in Art. 127, in which, putting
$$y = \tan x,$$
we found, in succession,
$$\begin{aligned} y_1 &= 1 + y^2, \\ y_2 &= 2y + 2y^3, \\ y_3 &= 2 + 8y^2 + 6y^4, \\ y_4 &= 16y + 40y^3 + 24y^5. \end{aligned}$$

Continuing in the same way, we find
$$\begin{aligned} y_5 &= 16 + 136y^2 + 240y^4 + \ldots, \\ y_6 &= 272y + \ldots, \\ y_7 &= 272 + \ldots. \end{aligned}$$

Hence, by putting $x=0$, and noticing that $y=0$ when $x=0$, we find
$$\begin{aligned} a_0 &= 0, \quad a_1 = 1, \\ a_2 &= 0, \quad a_3 = 2, \\ a_4 &= 0, \quad a_5 = 16, \\ a_6 &= 0, \quad a_7 = 272; \end{aligned}$$
$$\therefore \tan x = x + \frac{2x^3}{\underline{\lvert 3}} + \frac{16x^5}{\underline{\lvert 5}} + \frac{272x^7}{\underline{\lvert 7}} + \ldots.$$

Sometimes the process of forming the successive differential coefficients becomes very heavy and tedious, so that it would take a long time to find many of them. In some of such cases the difficulty may be avoided or diminished by special artifices. It must be remembered that we only require to find the successive derivatives for the sake of evaluating them when $x=0$; this will often enable us to save a good deal of labour. Some of the artifices which may be useful will be exemplified in the following articles.

163. Expansion by means of a differential equation. Very often the easiest way of obtaining the successive values of the coefficients in the expansion of a function y is to form an equation connecting y and its derivatives, and then, by Leibnitz' theorem, or otherwise, find the result of differentiating this relation n times. Then find the effect of putting $x=0$ in the resulting identity and we shall obtain an equation by which the terms of the expansion can be determined.

To return to the case of $\tan x$.

We found that $y_1=1+y^2$. This is a differential equation between y and y_1. If we differentiate both sides of this identity n times, using Leibnitz' theorem to differentiate the right-hand side, we find, as in Art. 135, p. 141,

$$y_{n+1}=yy_n+ny_1y_{n-1}+\frac{n(n-1)}{1.2}y_2y_{n-2}+\dots.$$

Hence, when $x=0$,

$$a_{n+1}=a_0a_n+na_1a_{n-1}+\frac{n(n-1)}{1.2}a_2a_{n-2}+\dots.$$

$$\begin{aligned}\text{Thus}\quad a_3&=a_0a_2+2a_1^2+a_2a_0=2a_0a_2+2a_1^2,\\ a_5&=a_0a_4+4a_1a_3+6a_2^2+4a_3a_1+a_4a_0=2(a_0a_4+4a_1a_3)+6a_2^2,\\ a_7&=2(a_0a_6+6a_1a_5+15a_2a_4)+20a_3^2,\\ &\dots\dots\dots\dots\dots\dots\end{aligned}$$

Also $a_1=1$ (from $y_1=1+y^2$).

If we notice that the expansion of $\tan x$ can contain only odd powers of x, since $\tan(-x)=-\tan x$ (see p. 49), we see that a_0, a_2, a_4, ... must be all zero.

$$\begin{aligned}\text{Hence}\quad a_3&=2a_1^2=2,\\ a_5&=8a_1a_3=16,\\ a_7&=12a_1a_5+20a_3^2=272.\end{aligned}$$

The student may in a similar way find the expansion of $\sec x$, using the method indicated on p. 142, and making use of the coefficients just found in expanding $\tan x$. To avoid confusion, he had better use b_0, b_1, ... for the $\sec x$ coefficients, leaving a_0, a_1, ... for those of $\tan x$. The general relation will be found to be

$$b_{n+1} = a_n b_0 + n a_{n-1} b_1 + \dots + n a_1 b_{n-1} + a_0 b_n,$$

from which the successive coefficients can be calculated. [See also Ex. 32, p. 187.]

As another example, find the expansion of $\sin^{-1}x$, where $\sin^{-1}x$ denotes the *smallest* angle whose sine is x.

Let $y = \sin^{-1}x$, then, as on p. 142, we find that

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} = n^2 y_n.$$

Now, put $x=0$, and we obtain $a_{n+2} = n^2 a_n$.

We know that a_0, a_2, a_4 are all zero, since $\sin^{-1}x$ is an odd function of x. Also $a_1 = 1$.

$$\begin{aligned} \therefore\ a_3 &= 1, \\ a_5 &= 3^2 a_3 = 3^2, \\ a_7 &= 5^2.a_5 = 5^2.3^2, \\ &\dots\dots\dots \end{aligned}$$

$$\therefore\ \sin^{-1}x = x + \frac{x^3}{\underline{|3}} + \frac{3^2x^5}{\underline{|5}} + \frac{5^2.3^2x^7}{\underline{|7}} + \dots.$$

Here the law of formation of the successive coefficients is simple and we can write down as many of them as we please without difficulty. This is in great contrast to the expansions of $\tan x$ and $\sec x$, where each coefficient had to be laboriously formed.

Different functions differ very much, and very unexpectedly, in thi respect.

We might have found the expansion of $\tan x$ to a few terms from th equation

$$y_1 = 1 + y^2$$

by assuming

$$y = A_1 x + A_3 x^3 + A_5 x^5 + \dots$$

(having no even powers, since $\tan x$ is an *odd* function of x),

whence

$$y_1 = A_1 + 3A_3x^2 + 5A_5x^4 + \dots.$$

Now, substituting these values in the above differential equation, w must have

$$A_1 + 3A_3x^2 + 5A_5x^4 + \dots \equiv 1 + (A_1x + A_3x^3 + A_5x^5 + \dots)^2.$$

Hence, equating coefficients,

$$\begin{aligned} A_1 &= 1, \\ 3A_3 &= A_1{}^2 = 1, \\ 5A_5 &= 2A_1A_3 = \tfrac{2}{3}, \\ 7A_7 &= A_3{}^2 + 2A_1A_5 = \tfrac{1}{9} + \tfrac{4}{15} = \tfrac{17}{45}, \end{aligned}$$

whence

$$\tan x = x + \tfrac{1}{3}x^3 + \tfrac{2}{15}x^5 + \tfrac{17}{315}x^7 + \dots.$$

The same method could have been employed also to determine t expansion of $\sin^{-1}x$.

In many physical investigations a differential equation is arrived at whose solution is required. In such cases we may find the solution in the form of a series by one of the methods indicated above. Examples will be found at the end of this chapter. In most cases it will be found that the series will involve one or more arbitrary constants.

164. Expansion by use of known series. Very frequently, when the direct use of Maclaurin's theorem would be very tedious, a few terms of an expansion may be readily found by making use of series already found.

Thus, to expand $e^{\sin x}$ in powers of x, to x^4 inclusive, we have

$$e^{\sin x} = 1 + \sin x + \tfrac{1}{2}\sin^2 x + \tfrac{1}{6}\sin^3 x + \tfrac{1}{24}\sin^4 x + \dots$$

and $\qquad \sin x = x - \tfrac{1}{6}x^3 + \dots;$

$$\begin{aligned}\therefore\ e^{\sin x} &= 1 + (x - \tfrac{1}{6}x^3) + \tfrac{1}{2}(x - \tfrac{1}{6}x^3)^2 + \tfrac{1}{6}(x - \dots)^3 + \tfrac{1}{24}(x - \dots)^4 + \dots \\ &= 1 + x - \tfrac{1}{6}x^3 + \tfrac{1}{2}(x^2 - \tfrac{1}{3}x^4) + \tfrac{1}{6}x^3 + \tfrac{1}{24}x^4 + \dots \\ &= 1 + x + \tfrac{1}{2}x^2 - \tfrac{1}{8}x^4 + \text{higher powers of } x.\end{aligned}$$

Other examples in which this method is useful will be found below.

EXAMPLES.

1. Find the first four terms in the expansion of $\sec\theta$.

2. Expand $\log\sec x$ to the term involving x^6 inclusive.

3. If $y = \sin^{-1}x = a_0 + a_1 x + a_2\dfrac{x^2}{1.2} + \dots,$

prove that $\qquad (1 - x^2)y_2 = xy_1,$

and deduce the expansion by the method of equating coefficients.

4. Write down the expansion of θ in powers of $\sin\theta$ [*i.e.* $\sin^{-1}x$ in powers of x].

5. If $y = \tan^{-1}x = a_0 + a_1 x + \dots,$

prove that $\qquad a_{n+2} = -n(n+1)a_n,$

and write down the expansion of $\tan^{-1}x$; *i.e.* express θ in a series of powers of $\tan\theta$.

6. Expand $e^{a\sin^{-1}x}$ in powers of x.

7. Expand $e^{ax}\sin bx$ in powers of x to the term involving x^3 inclusive.

8. Expand $e^{ax}\cos bx$ to the term involving x^3 inclusive.

9. Find the expansion of $\sin^2 x$ in powers of x.

10. Find the expansion of $\sin x \cos 2x$ in powers of x.

11. Find two terms of the expansion of $\log\left(\dfrac{\sin x}{x}\right)$.

12. Show that for small values of θ, the equation $\theta = \frac{1}{3}(\tan\theta + 2\sin\theta)$ is very accurate.

Test the equation when θ is the circular measure of 20°.

13. Show that $\theta = \frac{1}{3}(8\sin\frac{1}{2}\theta - \sin\theta)$
and $\theta = \frac{2}{3}(8\sin\frac{1}{4}\theta - \sin\frac{1}{2}\theta)$
are good approximations to θ, for small angles.

Test the formulae for an angle of 20°.

14. If c_1 is the chord of a circular arc, and c_2 is the chord of half the arc, show that, approximately,

$$\text{the arc} = \tfrac{1}{3}(8c_2 - c_1).$$

15. Find A, B, C so as to obtain the best approximation to the equation

$$\theta = \mathrm{A}\sin\theta + \mathrm{B}\sin 2\theta + \mathrm{C}\sin 4\theta,$$

when θ is small.

16. Expand $\lambda(\theta)$ in powers of θ up to θ^5.

$\left[\text{NOTE.}—\lambda(\theta) = \log(\sec\theta + \tan\theta) = \log\tan\left(\dfrac{\pi}{4} + \dfrac{\theta}{2}\right)\right]$.

17. Prove that $\lambda(\theta) = 2(\tan\frac{1}{2}\theta + \frac{1}{3}\tan^3\frac{1}{2}\theta + \frac{1}{5}\tan^5\frac{1}{2}\theta + \dots)$.

18. Prove that $\lambda(\theta) + \theta = 4(\tan\frac{1}{2}\theta + \frac{1}{5}\tan^5\frac{1}{2}\theta + \dots)$.

19. Test the approximate formula $\lambda(\theta) = 4\tan\frac{1}{2}\theta - \theta$ when θ is the circular measure of 20°.

20. Prove that $\lambda(\theta)$ is approximately $\frac{1}{3}(2\tan\theta + \sin\theta)$, and that $\frac{1}{3}(10\tan\frac{1}{2}\theta - 4\sin\frac{1}{2}\theta)$ is a better approximation.

Test the formulae for an angle of 20°.

21. Solve the differential equation $(1 - x^2)\dfrac{d^2y}{dx^2} = x\dfrac{dy}{dx}$ by assuming

$$y = \mathrm{A}_0 + \mathrm{A}_1 x + \mathrm{A}_2 x^2 + \dots.$$

22. Solve the differential equation $\dfrac{d^2y}{dx^2} + xy = 0$ by using Leibnitz' and Maclaurin's theorems.

[We shall find $a_2 = 0$, and $a_{n+3} = -(n+1)a_n$. These conditions will enable all the non-zero coefficients to be expressed in terms of a_0 and a_1, which will remain as two arbitrary constants.]

23. Solve the equation $x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + y = 0$.

24. Solve the equation $x\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}+xy=0$.

25. Solve the equation $\dfrac{d^2y}{dx^2}+n^2y=0$.

26. Prove that

$$f(a+h)-f(a-h)=2\left(hf'(a)+\frac{h^3}{\underline{|3}}f'''(a)+\frac{h^5}{\underline{|5}}f^{v}(a)+\dots\right).$$

and illustrate by expanding $\sin(\alpha+\beta)-\sin(\alpha-\beta)$ in powers of β.

27. If $f(x)$ is an algebraic function of x of degree not higher than the fourth, prove that

$$\tfrac{1}{3}h[f'(a+h)+4f'(a)+f'(a-h)]=f(a+h)-f(a-h).$$

28. Find the series for $\log_e(1+x)$ by writing down the expansion of $(1+x)^{-1}$ and integrating both sides of the equation.

29. Expand $(1-x^2)^{-\frac{1}{2}}$ by the Binomial Theorem, and, by integration, deduce the series for $\sin^{-1}x$.

30. Expand $(1+x^2)^{-1}$, and deduce the series for $\tan^{-1}x$.

31. If
$$y=\tan x=a_0+a_1x+a_2\frac{x^2}{\underline{|2}}+\dots,$$

by writing $y\cos x=\sin x$, and applying Leibnitz's theorem (see Ex. 10, p. 143), prove that

$$a_n-\frac{n(n-1)}{1.2}a_{n-2}+\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}a_{n-4}-\dots=\sin\frac{n\pi}{2},$$

and hence calculate the successive coefficients in the expansion.

32. If
$$\sec x=b_0+b_1x+b_2\frac{x^2}{\underline{|2}}+\dots,$$

prove that
$$b_n-\frac{n(n-1)}{1.2}b_{n-2}+\dots=0,$$

and hence calculate a number of the coefficients.

33. Expand $x\operatorname{cosec}x$ in powers of x, to x^6 inclusive, noting that $a_0=1$, and that the expression contains only even powers of x.

CHAPTER XII.

INDETERMINATE FORMS.

165. The indeterminate form $\frac{0}{0}$.

In evaluating a function of any variable x, it sometimes happens that for a particular value of x the function assumes an indeterminate form. For example. in evaluating the fraction

$$\frac{x^3-2x^2-5x+6}{x^2+x-2} \text{ for different values of } x,$$

it happens that if we try to insert $x=1$, the fraction assumes the form $\frac{0}{0}$, which is indeterminate. It is not the fraction itself which is indeterminate, but merely the particular form in which it is presented. In fact $x-1$ is a common factor of both numerator and denominator, and must be cancelled before we can evaluate the fraction for this particular value of x. Another common factor is $x+2$, so that the form of the fraction is also indeterminate when $x=-2$. If, however, we reduce the fraction to its lowest terms by cancelling the common factors, the form then obtained will not be indeterminate for any non-infinite value of x.

The object of the present chapter is to show various methods of evaluating a function when it assumes an indeterminate form, and more particularly when that form is $\frac{0}{0}$. In the present case nothing has to be done but to divide by the G.C.M. of the numerator and denominator, giving us the identity

$$\frac{x^3-2x^2-5x+6}{x^2+x-2}=x-3.$$

Hence the value of the fraction is -2 when $x=1$, and -5 when $x=-2$.

We have already had several important fractions which assume the form $\frac{0}{0}$ when $x=0$, viz.,

$$\frac{\sin x}{x}, \quad \frac{\tan x}{x}, \quad \frac{e^x-1}{x}, \quad \frac{a^x-1}{x},$$

whose values we have found to be respectively 1, 1, 1, $\log_e a$.

In all these cases the evaluation was, or could have been, effected by expressing the functions in powers of x, and then dividing by x. In fact, the fractions are not in their lowest terms, the numerator and denominator having the common factor x.

In general, if a fraction $\frac{f(x)}{\phi(x)}$ *becomes of the form* $\frac{0}{0}$ *when* $x=a$, *the fraction is not in its lowest terms, but either* $x-a$ *or some power of* $x-a$ *is common to both numerator and denominator, and must by some means be divided out before evaluation is possible for this value of* x.

This theorem follows at once from the following fundamental theorem:

166. If f(x)=0 when x=a, either x−a or some positive power of x−a is a factor of f(x).

This theorem may be taken as axiomatic, but it may be proved as follows in the case of functions which can be expanded in powers of $x-a$ by Taylor's Theorem.

Let $x-a=h$, so that $x=a+h$;

$$\therefore f(x)=f(a+h)=f(a)+hf'(a)+\frac{h^2}{1.2}f''(a)+\dots$$

$$=h\left[f'(a)+\frac{h}{1.2}f''(a)+\dots\right],$$

since by supposition, $f(a)=0$.

Hence h, which is $x-a$, is a factor of $f(x)$.

NOTE.—If $f(a)$ and $f'(a)$ are *both* zero, $(x-a)^2$ is a factor of $f(x)$, and so on.

167. Evaluation of the indeterminate form $\frac{0}{0}$.

Let u and v be two functions of x which both vanish when $x=a$. It is required to find the value of $\frac{u}{v}$ when $x=a$.

(1) *Division Method.* Evidently, if we can divide both numerator and denominator by $x-a$, or such power of $x-a$ as is common to them, the indeterminateness disappears. This is therefore one method of evaluation. This method is indeed the foundation of all the methods which can be adopted, though in many cases direct division is not practicable.

(2) *Differential Method.* A method which is often the easiest to adopt is obtained by considering the value of the fraction when x is just greater or less than a, and finding its limiting value as x approaches nearer and nearer, without limit, to the value a.

Now, when $x=a,\ u=0,$

and when $x=a+dx,\ u=0+du;$

similarly, when $x=a+dx,\ v=0+dv;$

$$\therefore \text{ the limiting value of } \frac{u}{v} \text{ is } \frac{du}{dv}.$$

For example, the value of $\frac{x^3-2x^2-5x+6}{x^2+x-2}$ when $x=1$

$$=\frac{0}{0}=\frac{(3x^2-4x-5)dx}{(2x+1)dx}=\frac{3x^2-4x-5}{2x+1}=-2.$$

Note, that we have here divided both numerator and denominator by dx, which is the value of $x-1$ when $x=1+dx$; consequently the essence of this method consists in cancelling the common factor $x-1$ and then evaluating the resulting fraction.

If the square of $x-1$ had been common to both numerator and denominator, the new fraction would have also assumed the form $\frac{0}{0}$, and could be evaluated by using the same method on it.

(3) *Expansion method.* In the differential method we assumed a value of x differing *infinitesimally* from its critical value a, and so obtained for numerator and denominator their *infinitesimal* increments from zero. These increments were all that were needed for our purpose, but were only partial increments, as we know that differentials are obtained by neglecting powers of dx higher than the first. We may, however, by help of Taylor's theorem, obtain the complete increments, in which case we need not in the first place assume the increase of x to be infinitesimal. Let $x=a+h$, and for clearness let $u=f(x)$ and $v=\phi(x)$.

Then

$$\frac{f(a+h)}{\phi(a+h)}=\frac{f(a)+hf'(a)+\frac{h^2}{\underline{|2}}f''(a)+\dots}{\phi(a)+h\phi'(a)+\frac{h^2}{\underline{|2}}\phi''(a)+\dots}=\frac{hf'(a)+\frac{h^2}{\underline{|2}}f''(a)+\dots}{h\phi'(a)+\frac{h^2}{\underline{|2}}\phi''(a)+\dots},$$

since, by supposition, both $f(a)$ and $\phi(a)=0$.

Now we can divide out the common factor h (*i.e.* $x-a$), so obtaining the equation

$$\frac{f(a+h)}{\phi(a+h)}=\frac{f'(a)+\frac{h}{\underline{|2}}f''(a)+\dots}{\phi'(a)+\frac{h}{\underline{|2}}\phi''(a)+\dots}.$$

Now, put $h=0$, and we find

$$\frac{f(a)}{\phi(a)}=\frac{f'(a)}{\phi'(a)}.$$

This is exactly the same result as is given by the second method.

If both $f'(a)$ and $\phi'(a)=0$, the expansion method enables us to at once divide out by h^2, *i.e.* $(x-a)^2$, obtaining finally in this case

$$\frac{f(a)}{\phi(a)}=\frac{f''(a)}{\phi''(a)}.$$

In fact, if a high power of $x-a$ is common to numerator and denominator, the expansion method enables us quickly to cancel the proper power, and evaluate the fraction.

Note that the form of any fraction is only indeterminate as long as *both* numerator and denominator vanish for the particular value of x. If the numerator $=0$, but not the denominator, the value of the fraction is zero; and if the denominator vanishes, but not the numerator, the value is infinite.

168. Note on the case when the value of the indeterminate form is zero or infinite.

In these two special cases, viz. when the true value of the indeterminate form $f(a) \div \phi(a)$ is zero or infinite, there is a peculiarity in the equation $\dfrac{f(a)}{\phi(a)} = \dfrac{f'(a)}{\phi'(a)}$ which, though of no importance practically, is yet sufficiently interesting to notice. The test of perfect equality of two expressions is two-fold, viz. that their ratio shall be unity, and their difference zero There is, however, an imperfect equality possible when the quantities themselves are zero or infinite, viz. that they are both zero or both infinite. We shall find that the above equation is of this imperfect nature. Of course if both fractions are zero, their difference is zero; we shall prove that their ratio is not unity but merely finite. When both are infinite we shall prove that their ratio is finite, but not unity, and consequently their difference must be infinite, so that in this case the equation satisfies neither the ratio test nor the difference test, but just merely the condition (which is all we here practically care about) that *both* are infinite if one is.

For suppose that $f(a), f'(a), \ldots, f^{r-1}(a)$ are all zero, but not $f^r(a)$, then by Taylor's theorem

$$f(a+h) = \frac{h^r}{\underline{|r}} f^r(a) + \ldots$$

and
$$f'(a+h) = \frac{h^{r-1}}{\underline{|r-1}} f^r(a) + \ldots$$

And suppose that $\phi(a)$, $\phi'(a)$, …, $\phi^{s-1}(a)$ are all zero, but not $\phi^s(a)$, so that

$$\phi(a+h)=\frac{h^s}{\underline{|s}}\phi^s(a)+\dots$$

and
$$\phi'(a+h)=\frac{h^{s-1}}{\underline{|s-1}}\phi^s(a)+\dots$$

……………………………

Then, by taking the limiting values of

$$\frac{f(a+h)}{\phi(a+h)} \text{ and of } \frac{f'(a+h)}{\phi'(a+h)},$$

we see that $\dfrac{f(a)}{\phi(a)}=0$ if $r>s$, and $\dfrac{f(a)}{\phi(a)}=\infty$ if $s>r$, and so is $\dfrac{f'(a)}{\phi'(a)}$; but the ratio $\dfrac{f(a)}{\phi(a)}\div\dfrac{f'(a)}{\phi'(a)}$ equals $\dfrac{s}{r}$, which is *finite* but not *unity*.

Similarly $\dfrac{f'(a)}{\phi'(a)}\div\dfrac{f''(a)}{\phi''(a)}=\dfrac{s-1}{r-1}$, and so on.

Hence, the fractions are all of the same *order* of magnitude, but are not equal in the full sense of the term.

Of course, this limited equality is sufficient. All we want to know in such case is that the fraction *is* zero, or infinite, as the case may be.

On the other hand, if $s=r$, we have the *perfect* series of equations

$$\frac{f(a)}{\phi(a)}=\frac{f'(a)}{\phi'(a)}=\dots=\frac{f^r(a)}{\phi^r(a)};$$

i.e. the equality is complete in the case in which complete equality is essential, viz. when the value of the fraction is finite; and though the equality is incomplete in other cases, it is sufficient for the purpose in view. [*N.B.*—Of course an equation between finite quantities must be complete: if two *finite* quantities are equal, their ratio must be unity, and their difference zero. It is only in the case of infinitesimal or infinite quantities that an imperfect equation is possible.]

EXAMPLES.

1. Prove that $\frac{\sin\theta}{\theta}=1$ when $\theta=0$, if θ is in radians.

2. Draw the curve $y=\frac{\sin x}{x}$ between the limits $x=\pm 1$ radian.

3. Evaluate $\frac{\tan\theta-\sin\theta}{\sin^3\theta}$ when $\theta=0$.

4. Evaluate $\frac{x^3+x^2-12}{x^4-5x^2+7x-10}$ when $x=2$.

5. Evaluate $\frac{x^4-4x^3+7x^2-12x+12}{x^3-x^2-8x+12}$ when $x=2$.

6. Evaluate $\frac{a^x-b^x}{x}$ when $x=0$.

7. Evaluate $\frac{x^n-a^n}{n}$ and $\frac{x^n-1}{n}$ when $n=0$.

8. Express $\frac{x^5-32y^5}{x^2-4y^2}$ in terms of x when $x=2y$.

9. Express $\frac{x^5+3x^4y+2xy^4}{x^4-2x^3y-3y^4}$ in terms of x when $x+y=0$.

10. Evaluate $\sqrt{\left(\frac{x^6-3x^2+2}{x^4-2x^3-3}\right)}$ when $x=-1$.

11. Evaluate $\frac{\sqrt{x^2-4}}{\sqrt{x}-\sqrt{2}+\sqrt{x-2}}$ when $x=2$.

12. Evaluate $\frac{\log x}{1-x}$ when $x=1$.

13. Evaluate $\frac{\sin 2x\tan^2 x}{(e^x-1)^3}$ when $x=0$.

14. Evaluate $\frac{e^x-e^{-x}-2x}{x-\sin x}$ when $x=0$.

15. Evaluate $\frac{\log\sin 2\theta}{\sin\theta+\cos\theta-\sqrt{2}}$ when $\theta=\frac{\pi}{4}$.

16. Evaluate $\frac{\cos nx-\cos x}{1-n^2}$ when $n=1$.

17. Evaluate $\frac{\sin ax-a\sin x}{\tan ax-a\tan x}$ when $a=1$.

169. Other indeterminate forms.

There are other indeterminate forms besides $\frac{0}{0}$, though this form is by far the most important, and the evaluation of the others depends sooner or later on the evaluation of this form.

The other forms are

(2) $\frac{\infty}{\infty}$, $\left\{\text{such as } \frac{\sec\theta}{\tan\theta} \text{ when } \theta=\frac{\pi}{2}, \text{ or } \frac{\log x}{x} \text{ when } x=\infty;\right.$

(3) $\infty-\infty$, $\left\{\text{such as } \sec\theta-\tan\theta, \text{ or } \sec^n\theta-\tan^n\theta, \text{ when } \theta=\frac{\pi}{2};\right.$

(4) $0\times\infty$, as, for example, $x\log x$ when $x=0$;

and the exponential forms

(5) 1^∞, $\left\{\text{such as } \left(1+\frac{1}{p}\right)^p \text{ when } p=\infty, \text{ or } (1+x)^{\frac{1}{x}} \text{ when } x=0;\right.$

(6) 0^0, such as x^x, or $x^{\sin x}$, when $x=0$;

(7) ∞^0, $\left\{\text{such as } (\tan x)^{\cos x}, \text{ when } x=\frac{\pi}{2}.\right.$

The general method of solving cases (2), (3), and (4) consists in reducing them to the standard form $\frac{0}{0}$, and then making use of one of the methods already explained, though in some cases we may deal with these indeterminates in the form $\frac{\infty}{\infty}$ by the method of next article before reducing to the standard form.

In the cases (5), (6), and (7) their *logarithms* are all of the form $0\times\infty$, and can therefore be evaluated as indicated above.

Thus, if $u=(1+x)^{\frac{1}{x}}$, $\log_e u=\frac{\log(1+x)}{x}$, which is of the form $\frac{0}{0}$ when $x=0$. The limiting value of this fraction is unity when $x=0$.

Hence $u=e$, *i.e.* $(1+x)^{\frac{1}{x}}=e$ when $x=0$.

170. The form $\frac{\infty}{\infty}$.

When the numerator and denominator of a fractional function of x both become infinite for a particular value of x, the form of the function is indeterminate.

In such cases it may be possible to reduce the fraction to the standard form $\frac{0}{0}$, and so evaluate by methods already explained.

For example $\frac{\sec 3x}{\sec x}$ becomes $\frac{\infty}{\infty}$ when $x=\frac{\pi}{2}$.

We can write this fraction in the form $\frac{\cos x}{\cos 3x}$ which becomes $\frac{0}{0}$ for the required value of x, and on evaluation we find its value to be $-\frac{1}{3}$.

Sometimes, however, this transposition may be inconvenient, and the following theorem is useful.

If $\frac{u}{v}$ becomes $\frac{\infty}{\infty}$ when $x=a$, it may be replaced by $\frac{du}{dv}$ for this value of x.

For
$$\frac{u}{v}=\frac{\frac{1}{v}}{\frac{1}{u}}=\frac{-\frac{1}{v^2}dv}{-\frac{1}{u^2}du} \text{ when } x=a,$$

since the transformed fraction is of the form $\frac{0}{0}$.

Hence
$$\frac{u}{v}=\frac{u^2}{v^2}\cdot\frac{dv}{du},$$

i.e. finally,
$$\frac{u}{v}=\frac{du}{dv}.$$

For example, $\dfrac{\log\cos x}{\log\left(\frac{\pi}{2}-x\right)}$, when $x=\frac{\pi}{2}$,

$$=\frac{\infty}{\infty}=\frac{-\tan x\,dx}{-\dfrac{dx}{\frac{\pi}{2}-x}}=\frac{\frac{\pi}{2}-x}{\cot x}=\frac{0}{0}=\frac{-dx}{-\operatorname{cosec}^2 x.dx}$$

$$=\sin^2 x=1.$$

It is important to notice, however, that before we can succeed in evaluating the fraction, either it or one of its equivalent fractions must as a rule be brought into the form $\frac{0}{0}$ The only exception to this necessity is when the critical value of x is infinite. In such cases the differential method will often be successful without such transposition.

Thus, $\frac{\log x}{x}$, when $x=\infty$, $=\frac{\frac{1}{x}dx}{dx}=\frac{1}{x}=0.$

171. If, in the process of evaluating an indeterminate form, one of the factors is finite, the value of this factor may be inserted at once. This will often save considerable labour.

Thus, to find $y=\frac{\sec^3\theta-\tan^3\theta}{\tan\theta}$ when $\theta=\frac{\pi}{2}$.

Here
$$\begin{aligned} y&=\frac{1-\sin^3\theta}{\sin\theta\cos^2\theta}\\ &=\frac{1-\sin^3\theta}{\cos^2\theta}, \text{ since } \sin\frac{\pi}{2}=1,\\ &=\frac{-3\sin^2\theta\cos\theta\,d\theta}{-2\cos\theta\sin\theta\,d\theta}\\ &=\frac{3}{2}.\end{aligned}$$

EXAMPLES.

1. Evaluate $\frac{\sec x}{\tan x}$ and $\sec x-\tan x$, when $x=\frac{\pi}{2}$.

2. Evaluate $\sec^3 x-\tan^3 x$ when $x=\frac{\pi}{2}$.

3. Evaluate $x\log x$ when $x=0$.

4. Evaluate $\frac{e^x}{x^n}$ when $x=\infty$.

5. Evaluate $\frac{\sec^n x-\tan^n x}{\tan^{n-2}x}$ when $x=\frac{\pi}{2}$.

6. Evaluate $\tan x(\sec x-\tan x)$ when $x=\frac{\pi}{2}$.

7. Evaluate, when $x=0$, and also when $x=\frac{\pi}{2}$,

(i) $(1+\sin x)^{\cot x}$; (ii) $(1-\sin x)^{\cot x}$; (iii) $(\cos x)^{\cot x}$; (iv) $(\sin x)^{\tan x}$.

8. Evaluate $(1+\cos^2 x)^{\sec x}$ when $x=\frac{\pi}{2}$.

9. Evaluate $\left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}}$ and $\left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}}$ when $x=0$.

10. Evaluate $(\log x)^{\frac{1}{x}}$ when $x=\infty$.

11. Evaluate $(3-x)\tan\frac{x\pi}{6}$ when $x=3$.

12. Evaluate $\frac{(1-x^2)\log(1+x^2)}{\cos x . \log \sec x}$ when $x=0$.

172. Evaluation of a rational algebraic fraction $\frac{f(x)}{\phi(x)}$ when x is infinite.

This special case, which is very important, is also very simple, and does not need any of the above methods, the value of such a fraction reducing to the ratio of the highest term in the numerator to the highest term in the denominator, since the other terms are unimportant compared with the highest term when x is infinite.

Thus
$$\frac{2x^3-3x^2+x-5}{3x^3+5x^2-x+6}=\frac{2}{3}$$
when x is infinite.

This is too obvious to require formal proof, but such proof can be supplied by putting $z=\frac{1}{x}$ and multiplying both numerator and denominator by z^3, bringing the fraction into the form
$$\frac{2-3z+z^2-5z^3}{3+5z-z^2+6z^3}$$
which $=\frac{2}{3}$ when $z=0$, *i.e.*, when $x=\infty$.

It is obvious that the value of such a fraction is finite only when the highest terms in the numerator and denominator are of the same degree. If the numerator is of higher degree than the denominator the fraction is infinite, and if the numerator is of lower degree the fraction is zero.

NOTE.—It may be noted that the differential method (Art. 170) is *legitimate* here, and would lead to the right result;

but the object of the present article is to show that such method is *unnecessary.*

173. Relative values of functions of two related variables when the variables are infinite.

It is frequently necessary, in the case of a fraction whose numerator and denominator are given as functions of two variables x and y between which there is a given relation, to ascertain the value of the fraction when the variables are infinite.

Such fractions present themselves for evaluation in connection with the problem of finding rectilinear and curvilinear asymptotes to a curve, and it is for this reason that they are od d here.

174. We shall, in this article, consider fractions in which x and y are connected by a linear equation, $y=mx+c$, and then take other less simple relations. The theory is very simple, there are some interesting points worthy of consideration.

The first example will be the ratio of two homogeneous functions of equal degree.

Thus: evaluate $\dfrac{x^2-3xy+2y^2}{2x^2+xy-y^2}$ when x and y are infinite, being given that $y=mx+c$.

Substituting $y=mx+c$ in the fraction, it becomes

$$\frac{(1-3m+2m^2)\,x^2+(4m-3)\,cx+2c^2}{(2+m-m^2)\,x^2+(1-2m)\,cx-c^2}=\frac{1-3m+2m^2}{2+m-m^2}$$

in the limit.

If $m=1$ or $\frac{1}{2}$ the value is zero, since the coefficient of x^2 in the numerator vanishes; and if $m=-1$ or 2 the value is infinite, since the coefficient of x^2 in the denominator vanishes.

In all other cases the ratio is finite.

It is very useful to notice that the value of c has no effect on the value of the fraction, and that it would have been just the same if the relation between y and x had been $y=mx$, *i.e.*, if c had been zero.

This is because c, being finite, is unimportant compared with x which is infinite. In fact, although the *difference* between

y and mx is not zero, the *ratio* of y to mx is unity, in the limit when x is infinite.

Evidently the ratio of y to mx is all that matters in evaluating the fraction, so that we may take any values of x and y that satisfy the ratio equation $\frac{y}{mx}=1$.

Hence, if we put $x=1$, $y=m$, we obtain the value of the fraction, and with the minimum of trouble.

This simple method of evaluation is of great use, but we must remember that it applies only to the ratio of two homogeneous functions of equal degree.

If the numerator is of higher degree than the denominator, the fraction is in general infinite when $x=\infty$, but an exception occurs when the relation between x and y is such as to make the highest power of x vanish in the numerator.

Thus: *To evaluate* $\left.\frac{x^3-3x^2y+2xy^2}{2x^2+xy-y^2}\right]_{x=\infty}$ when $y=x+c$.

Here $y-x$ is a factor of the numerator, the other factor being $2xy-x^2$.

$$\therefore \text{ the fraction} = (y-x)\left[\frac{2xy-x^2}{2x^2+xy-y^2}\right]=\tfrac{1}{2}c,$$

the value of $\frac{2xy-x^2}{2x^2+xy-y^2}$ $(=\frac{1}{2})$ being obtained at once by putting $x=1$, $y=1$ in accordance with the method of last article.

Similarly, if the numerator is of lower degree than the denominator, the fraction is in general zero when $x=\infty$, but an exception occurs when the relation between x and y is such as to make the highest power of x vanish in the denominator.

If the numerator is not homogeneous, the fraction can be broken up into a number of separate fractions in each of which the numerator is homogeneous.

If the denominator is not homogeneous, the terms of highest degree are the only important ones unless the relation $y=mx+c$ is such as to make them vanish when the value of y is substituted, *i.e.* unless $y-mx$ is a factor of the highest degree terms. In such a case the terms of next lower degree become equally important.

The whole theory, in fact, depends on the relative importance of the different terms when x is infinite.

The easiest way of discovering whether $y-mx$ is a factor of a homogeneous expression is by finding whether the expression does or does not become zero when we substitute $x=1$, $y=m$. If it does, $y-mx$ is a factor.

EXAMPLES.

Evaluate for infinite values of x and y, or of one of them:

1. $\dfrac{2ay^2}{x(x+y)}$ when $x=y+c$.

2. $\dfrac{bx^2-ay^2}{xy}$ when $x=y+c$.

3. $\dfrac{2x^3+7x^2y+5xy^2-2y^3}{3x^2-5xy+y^2}$ when $x+2y=c$.

4. $\dfrac{4xy^2-6y^3+3x^2-5xy+4}{2x^3-3x^2y+3y^2-5x+2}$ when $2x-3y=c$.

5. $\dfrac{2x^3+3x^2y-y^3+4x^2+2xy-2y^2}{4x^2+5xy+y^2}$ when $x+y=c$,

and also when $2x-y=c$.

6. $\dfrac{x^4-3xy^2+y^2}{2y^2-3x+y}$ when $x=c$.

175. Orders of infinite quantities.

We have seen that when $y=mx+c$, the evaluation of a fraction consisting of the ratio of two functions of x and y when x and y are infinite is effected by attending only to the terms of highest degree in both numerator and denominator unless $y-mx$ is a factor of such terms, in which case the terms of next lower degree become of equal importance.

We may express this systematically by saying that, if x and y are infinities of the first order,

the sum of the terms of the first degree in any expression is an infinity of the first order unless $y-mx$ is a factor, in which case the sum is finite;

the sum of the terms of the second degree is generally an infinity of the second order, but, if $y-mx$ is a factor, the sum is an infinity of the first order, and if $(y-mx)^2$ is a factor, the sum is finite, and so on, the general result being expressed by the statement that the sum of the terms of the n^{th} degree is an infinity of the n^{th} order unless some power of $y-mx$ is a factor, and if $(y-mx)^r$ is a factor, the sum is an infinity of the $(n-r)^{\text{th}}$ order.

With this understanding we may say that the ratio of two expressions is obtained by taking the ratio of the sum of the terms of highest order in each, neglecting all those of lower order; and that the ratio is finite, if the order of the sum of the highest terms in the numerator and denominator are equal, or, as we may say more concisely, if the numerator and denominator are of the same order of infinity;

the ratio is ∞, if the numerator is of a higher order of infinity than the denominator; and

the ratio is zero, if the numerator is of a lower order of infinity than the denominator.

NOTE.—The order of any *single* term in an expression is equal to its degree, but the *sum* of a number of terms of any degree may be of lower order by reason of some power of $y-mx$ occurring as a factor.

176. In the above articles we have supposed a linear relation to exist between x and y, so that x and y are of the same order of infinity: except in the special case when $m=0$ or ∞, in which case they are not of the same order, and usually one of the variables is finite while the other is infinite.

It is important to consider the case in which x and y are not of the same order of infinity, as such cases occur in connection with curvilinear asymptotes. We will suppose that, when x and y are infinite, x^A is proportional to y^B, *i e.*, x^A and y^B are of the same order of infinity. The problem will be to arrange the various terms $x^m y^n$ of any expression in order of magnitude when x and y are infinite. If A and B are unequal, it will no longer be true that the degrees of the various terms give their relative order. It may happen that xy^3 is of a higher order of infinity than x^3y^2, though of course such terms as x^2y^2 or x^2y will be of a lower order of infinity than x^3y^2 whatever the relation between x and y, as long as x and y become infinite together.

Example. If $x^2 = ay$, where a is finite, the terms x^3y^2 and xy^3 are of the same order of magnitude, being both comparable with x^7, the importance of the terms being estimated by taking the index of x + twice the index of y, and associating together all those terms for which this quantity is the same.

Generally, if x^A is of the same order of infinity as y^B, all terms x^my^n for which $Bm+An$ is the same are of the same order of infinity.

To fix the scale of infinities most conveniently, we will introduce a new variable z such that x^A and y^B are each comparable with z^{AB} and we will take the different powers of z as measuring the different orders of infinity of the different terms.

Then x is of the same order as z^B, *i.e.*, is of order B,
y is of the same order as z^A, *i.e.*, is of order A,
and x^my^n is of the same order as z^{Bm+An}, *i.e.*, is of order $Bm+An$,

i.e., the relative importance of the various terms x^my^n in any expression will be measured by the value of $Bm+An$ in each.

177. Relative values of functions of two related variables when the variables are infinitesimal.

Another question of equal importance with that of discovering the relative importance of terms when x and y are great, is that of finding their relative importance when x and y are very small. As the two methods of investigation are intimately connected, we will now consider the latter question.

If x and y are infinitesimal, and x^A is comparable with y^B, the different terms x^my^n will now be infinitesimals of different orders. Introducing z as before, and taking the powers of z as standards, we find

x is an infinitesimal of order B,
y .. .. .. A,
x^my^n ,, ,, ,, $Bm+An$.

Now infinitesimals of lower order are more important than those of higher order, so that here the *smaller* the value of $Bm+An$ the *greater* will be the importance of x^my^n. Hence the order of relative importance when x and y are very small, is the exact reverse of the order when x and y are very great.

EXAMPLES.

1. If x^2 is comparable with y, arrange the terms x^6, x^3y, y^3, xy^2 in order of relative importance

(1) when x and y are very great,

(2) when x and y are very small.

2. If x^2 is comparable with y^3, arrange the terms x^5, x^4y, x^3y^3, xy^5 in order of magnitude

(1) when x and y are very great,

(2) when x and y are very small.

3. If x^A is comparable with y^B, both being infinitesimal, show that $\frac{y}{x}$ is zero, finite, or infinite according as A is greater than, equal to, or less than B; and that the reverse is true if x and y are very great.

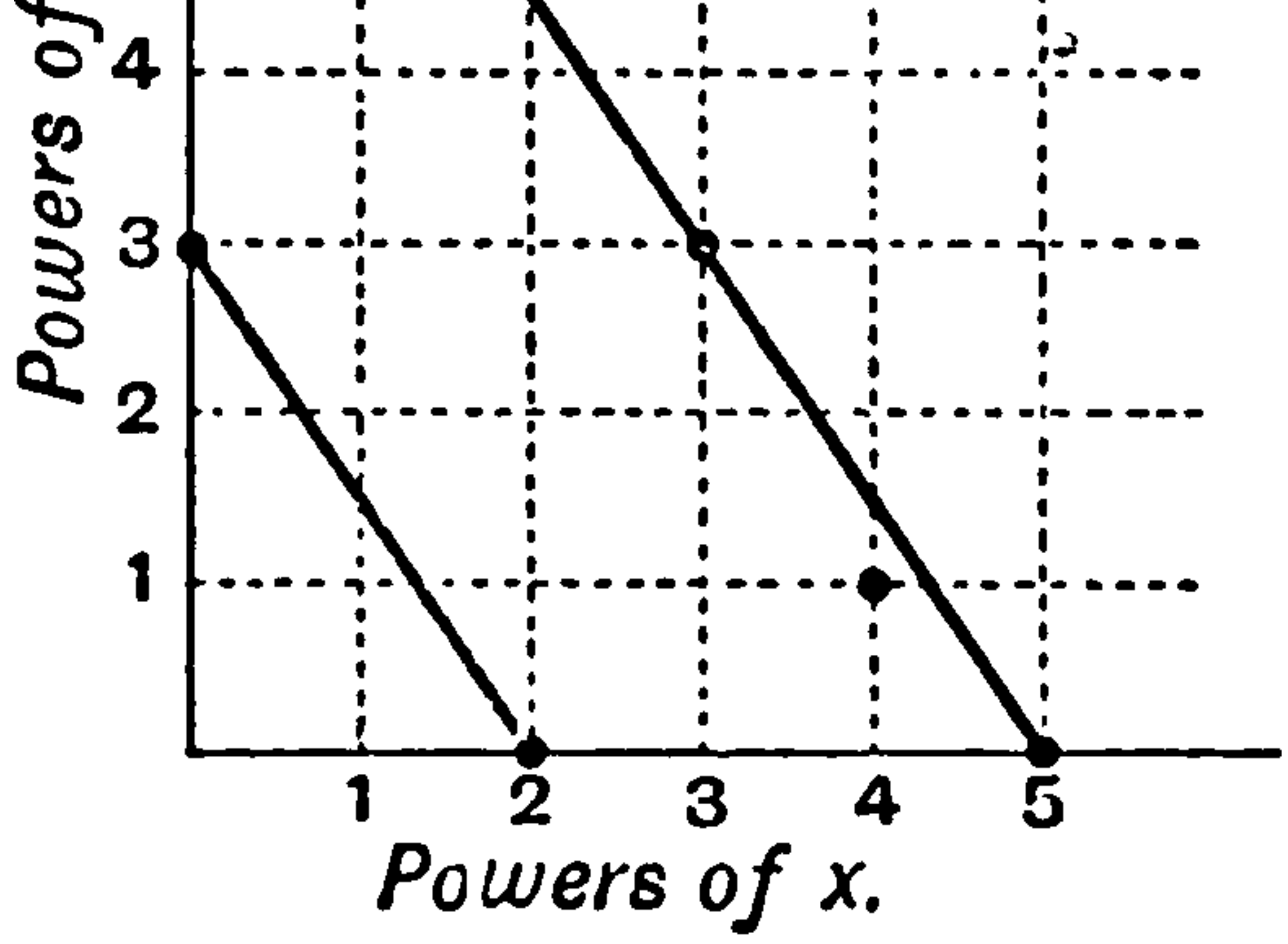
178. Graphic method of determining the relative importance of terms of the form $x^m y^n$ when x^A is comparable with y^B, when x and y are very great or very small.

We have seen that all terms whose indices are connected by the relation

$$Bm + An = \text{constant}$$

are of the same order of magnitude.

If, then, we represent the terms by points whose abscissae are the values of m, and whose ordinates are the values of n, in the various terms, the terms of equal importance will lie on the straight line $Bm + An = \text{constant}$.



The diagram illustrates the relative importance of various terms when x^2 is proportional to y^3. It shows the terms x^2 and y^3 connected by a line and the terms x^5, x^3y^3, xy^6 connected by a parallel line. It shows also the terms x^4y and xy^5, which are within the outer line, but x^4y is nearer to it than xy^5 is; hence they are both of lower order than any of those connected by the outer line, but x^4y is of higher order than xy^5. Hence when x and y are very great, the terms connected by the outer line are of greatest importance, and those nearer to it are more important than those more remote. If, on the other hand, x and y are infinitesimal, the terms connected by the lower line are the most important, and those nearer to it are more important than those more remote. We shall refer to this again in connection with curve tracing.

EXAMPLES.

1. Write all the terms in an expression of the 4th degree in order of importance when x and y are very large:

(1) when $y^2=ax$, (2) when $ay^2=x^3$,
(3) when $ay^3=x^4$, (4) when $y^4=a^3x$,

a being finite in each case.

2. Is it possible that x^3 and xy^3 can be the most important terms in the expression

$$Ax^3+Bx^2y+Cx^2y^2+Dxy^3$$

when both x and y are either very great or very small?

Which pair of terms can be of greatest importance in either case?

3. If $$2y^2+3xy=x^3-x^2+7,$$
which terms can be of greatest importance? Arrange the other terms in order.

4. If $y^2:x=3$, when x and y are infinite, find the approximate value of $\frac{6xy-4y^2}{3x+2y}$ when x and y are very great.

5. Find the approximate value of $\frac{9x^2+4y^2}{3x+2y}$ under the same circumstances.

[The neatest way of solving (4) and (5) is by ordinary division, continued as far as the term in the quotient which involves $y^2\div x$.]

CHAPTER XIII.

PARTIAL DIFFERENTIAL COEFFICIENTS.

179. Differentiation of a function of two variables. Partial Differential Coefficients.

When we have to differentiate a function of two variables, x and y, the most systematic way is to arrange the terms of the differential in two groups, according as they are multiples of dx or of dy.

Thus, if

$$u = 3x^2 - 5xy + 2y^2 - 15x + 6y - 8,$$

instead of writing

$$du = 6x dx - 5(y dx + x dy) + 4y dy - 15 dx + 6 dy,$$

and then arranging the terms into the two groups

$$du = (6x - 5y - 15) dx + (-5x + 4y + 6) dy,$$

we can, at the first, arrange them in these groups by first differentiating u throughout as if y was a constant, and then differentiating throughout as if x was a constant, the complete differential being the sum of these partial differentials. It is obvious that we may always do this, for every term of the differential must have dx or dy as a factor, and no term can have both, because $dx.dy$ is an infinitesimal of the second order.

The coefficient of dx in the above groupings is called the partial differential coefficient of u with regard to x, being evidently what the complete differential coefficient would be if x were the only variable; similarly the coefficient of dy is called the partial differential coefficient of u with respect to y.

The notation adopted for these differential coefficients is $\left(\frac{du}{dx}\right)^*$ and $\left(\frac{du}{dy}\right)$, so that the differential equation runs

$$du = \left(\frac{du}{dx}\right)dx + \left(\frac{du}{dy}\right)dy.$$

Similarly, in the case of a function of three or more variables, $x, y, z, \ldots,$

$$du = \left(\frac{du}{dx}\right)dx + \left(\frac{du}{dy}\right)dy + \left(\frac{du}{dz}\right)dz + \ldots.$$

180. Case when x and y are independent variables.

If x and y are independent of each other, the ratio of du to dx or to dy is an indeterminate one, inasmuch as the increments dx and dy may themselves have any ratio whatever, Thus, if x and y are the coordinates of a point in a plan of a district, and u is the elevation of the actual point above (say) sea-level, then du is the increase in elevation due to a slight movement away from the point, and this change evidently in general depends on the direction of movement, *i.e.* on the relative values of dx and dy. Thus, if the point is on the side of a hill, one direction (backwards or forwards) would cause no change of level, while any other would involve a more or less rapid alteration of altitude. We shall refer to this later. In Physics and Mechanics also, a quantity may be a function of several variables, some of which may vary independently of the others. Evidently the partial differential coefficients are the rates of change of the function when one variable changes and not the others.

The student should practise forming these partial differential coefficients.

N.B.—It is *unnecessary* in this case to put the brackets round the partial differential coefficients, and it is quite usual to omit them, as is done in the following examples.

EXAMPLES.

1. If $u = \frac{x^2 - 3xy - y^2}{x+y}$, write down $\frac{du}{dx}$ and $\frac{du}{dy}$.

2. If $u = a^x$, write down $\frac{du}{dx}$ and $\frac{du}{da}$.

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3. If $u = x^p y^q$, write down $\dfrac{du}{dx}, \dfrac{du}{dy}, \dfrac{du}{dp}$, and $\dfrac{du}{dq}$.

4. If $u = x^2 - 3xy + 2y^2 - 5ax + 6ay - a^2$, write down $\dfrac{du}{dx}, \dfrac{du}{dy}$, and $\dfrac{du}{da}$.

5. If $u = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$, show that $x\dfrac{du}{dx} + y\dfrac{du}{dy} + a\dfrac{du}{da} + b\dfrac{du}{db} = 0$.

6. In Example 1, show that $x\dfrac{du}{dx} + y\dfrac{du}{dy} = u$.

181. Case when y is a function of x. Distinction between total and partial differential coefficient.

If, however, y is a function of x, as, for example, if

$$u = (x^2 - 3x)y,$$

where $$y = \sin x,$$

then $$du = y(2x - 3)dx + (x^2 - 3x)dy,$$

whence $$\frac{du}{dx} = y(2x - 3) + (x^2 - 3x)\frac{dy}{dx}$$

$$= (2x - 3)\sin x + (x^2 - 3x)\cos x.$$

Here we have $du = \left(\dfrac{du}{dx}\right)dx + \left(\dfrac{du}{dy}\right)dy$,

and $$\frac{du}{dx} = \left(\frac{du}{dx}\right) + \left(\frac{du}{dy}\right)\frac{dy}{dx};$$

and it becomes important to thoroughly realize the distinction between the complete differential coefficient $\dfrac{du}{dx}$ and the partial one $\left(\dfrac{du}{dx}\right)$. The bracket round the partial differential coefficient is an excellent way of distinguishing it. Another method will be given below, which is often advantageous.

As a matter of fact, when y is a function of x, any function u, of x, might be expressed as a function of x and y in very different ways; and the value of the partial differential coefficients would depend on the way in which y was introduced. Thus, taking another example:

If $y = \sin x$, and $u = x^2 \sin x \cos x$,

we might write $u = x^2 y \cos x$,

or $u = x^2 y \sqrt{1 - y^2}$,

or $u = (\sin^{-1} y)^2 \sin x \cos x$,

or in many other ways, in each of which $\left(\dfrac{du}{dx}\right)$ would be different, and so would $\left(\dfrac{du}{dy}\right)$, though of course $\dfrac{du}{dx}$ would be the same in every case.

It is better in such cases to adopt a notation which recognizes that the partial differential coefficients depend on the form presented by the function, and we might say, if $u = f(x, y)$,

$$du = \left(\frac{df}{dx}\right) dx + \left(\frac{df}{dy}\right) dy,$$

and
$$\frac{du}{dx} = \frac{df}{dx} + \frac{df}{dy} \cdot \frac{dy}{dx},$$

where $f(x, y)$ denotes not only the function, but the *form* in which it is given, and $\dfrac{df}{dx}$ and $\dfrac{df}{dy}$ obviously can then denote only the *partial* differential coefficients, so that the brackets round them are not absolutely necessary, and can be used or omitted as desired.

If, instead of y being given as a function of x, both x and y are given functions of some third variable, t, the equation may be written

$$\frac{du}{dt} = \left(\frac{df}{dx}\right) \frac{dx}{dt} + \left(\frac{df}{dy}\right) \frac{dy}{dt}.$$

N.B.—The brackets, though not absolutely necessary, are useful to emphasize the distinction between partial and complete differential coefficients.

Similarly, if we are dealing with a function $u = f(x, y, z, \ldots)$ of more than two variables, we shall have

$$\frac{du}{dt} = \left(\frac{df}{dx}\right) \frac{dx}{dt} + \left(\frac{df}{dy}\right) \frac{dy}{dt} + \left(\frac{df}{dz}\right) \frac{dz}{dt} + \ldots.$$

182. Gradient at any point of the curve f(x, y) = constant. There is another way in which y may be determined as a function of x, viz. when the given function of x and y has a constant value; *e.g.* if $u=0$ is the equation of a curve, x, y being the coordinates of a point on the curve. In such case $u=0$ is itself the equation giving the relation between x and y. In such case the partial differential coefficients $\left(\frac{du}{dx}\right)$ and $\left(\frac{du}{dy}\right)$ have no ambiguity, but are perfectly definite in form. This is in fact the kind of case in which we most frequently require to use these derived functions.

Since u is constant in value, the complete differential coefficient $\frac{du}{dx}$ is zero, and therefore the symbol for it is not required; hence we may, if we like, omit the brackets from the partial differential coefficients. In what follows we shall sometimes use the brackets and sometimes omit them.

If we differentiate $u=0$, we obtain

$$\left(\frac{du}{dx}\right)dx+\left(\frac{du}{dy}\right)dy=0,$$

and thus obtain the gradient of the curve at any point (x, y), viz.

$$\frac{dy}{dx}=-\frac{\frac{du}{dx}}{\frac{du}{dy}}.$$

E.g. If the curve is $3x^2-5xy+2y^2-15x+6y-8=0$, we find

$$\frac{dy}{dx}=-\frac{6x-5y-15}{-5x+4y+6},$$

which gives us the gradient for any assigned point of the curve.

It is rather interesting to see what would be given by the gradient equation if we inserted the coordinates of a point which is not on the curve. This may be left to the student, with the remark that $u=0$ is only one of a whole family of curves, $u=$ constant, which have the same gradient equation.

Example. To what curve of the above family of curves does the point (1, 1) belong, and what is the gradient of that curve at this point?

Ans. $3x^2-5xy+2y^2-15x+6y+9=0$; $2\frac{4}{5}$.

183. Meaning of the equations $\left(\frac{du}{dx}\right)=0$, $\left(\frac{du}{dy}\right)=0$.

At points on the curve at which $\frac{du}{dx}=0$ the gradient of $u=0$ is zero.

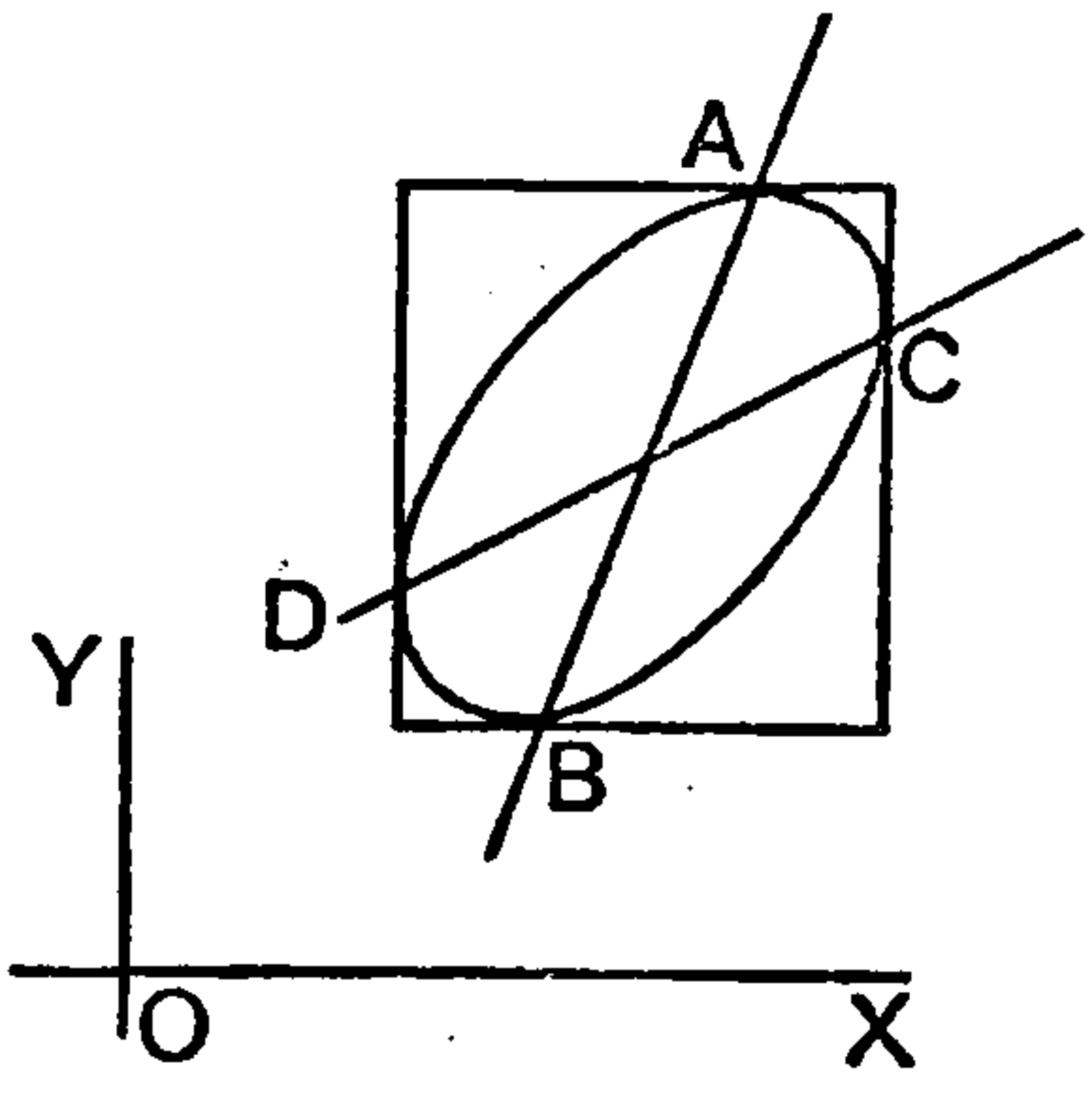
Hence $\frac{du}{dx}=0$ is the equation of a locus which goes through all the points on the given curve where the gradient $=0$.

Moreover, if we consider the whole series of curves given by the equation $u=$ constant, the locus given by the equation $\frac{du}{dx}=0$ cuts the whole series in points of zero gradient, *i.e.* it is the locus of points of zero gradient in the given family of curves. Similarly $\frac{du}{dy}=0$ is the locus of points of infinite gradient.

If at any point on the curve *both* $\frac{du}{dx}$ and $\frac{du}{dy}$ are zero, the expression for the gradient becomes indeterminate. Such a point is called a singular point on the curve. It may be an isolated point (called a *conjugate* point), or there may be two branches of the curve passing through it, in which case it is called a double point. (See Arts. 188, 194, 196.)

Application to conic sections.

If $u=0$ is a conic section, *i.e.* is of second degree, $\frac{du}{dx}=0$ is a straight line going through the points of zero gradient on the conic, *i.e.* through the points where y is a maximum or a minimum. Thus, in the figure, the line AB is given by the equation $\frac{du}{dx}=0$.



Since the tangents at A and B are parallel, it follows that $\frac{du}{dx}=0$ is a *diameter*. Similarly $\frac{du}{dy}=0$ is the diameter which goes through the points of infinite gradient, and therefore the centre of the conic is given by the point of intersection of these two lines.

The general equation to a diameter is $\left(\frac{du}{dx}\right)+\lambda\left(\frac{du}{dy}\right)=0$, where λ is any constant.

For example, the centre of the conic $3x^2-5xy+2y^2-15x+6y-8=0$ is given by the intersection of the diameters

$$6x-5y-15=0,$$
$$-5x+4y+6=0.$$

Any other diameter can be found by treating these simultaneously, *e.g.* $x-y-9=0$ is a diameter, and so are the lines

$$y+39=0 \text{ and } x+30=0.$$

Hence the centre is at the point $(-30, -39)$.

Since the tangents at the ends of $6x-5y-15=0$ are parallel to the axis of x, it follows that

$$6x-5y-15=0$$

and

$$y+39=0$$

are a pair of conjugate diameters.

Similarly and $\left.\begin{array}{r}5x-4y-6=0\\ x+30=0\end{array}\right\}$ are conjugate diameters.

The equation of the curve may be written in the form

$$(6x-5y-15)^2-(y+39)^2+1200=0,$$

showing that the curve is a hyperbola whose asymptotes are

$$6x-5y-15\pm(y+39)=0,$$

i.e. and $\left.\begin{array}{r}6x-4y+24=0\\ 6x-6y-54=0,\end{array}\right\}$

i.e. and $\left.\begin{array}{r}3x-2y+12=0\\ x-y-9=0.\end{array}\right\}$

The curve has a pair of horizontal tangents given by

$$y+39=\pm\sqrt{1200}=\pm 20\sqrt{3}=\pm 34{\cdot}64.$$

From these data the curve can be drawn.

EXAMPLES.

1. Find the centres of the following curves:

(i) $4x^2+12xy+10y^2-4x-4y-2=0$;

(ii) $3x^2-12xy+9y^2+6x-6y-4=0$;

(iii) $xy-3x+2y-5=0$.

2. Show that the centre of the curve $x^2-4xy+4y^2-2x+3y=0$ is at an infinite distance, and draw the curve.

3. If $u \equiv x^2 - 4xy + 3y^2 + 2ax - 2ay$, draw the lines given by the equations $\frac{du}{dx}=0$, $\frac{du}{dy}=0$, and $\frac{du}{da}=0$. What locus is represented by the equation $u=0$?

4. If $u \equiv x^3 + y^3 - 3axy$, draw the loci represented by $\frac{du}{dx}=0$ and $\frac{du}{dy}=0$, and find where they cut the curve $u=0$.

184. Homogeneous Functions. Euler's Theorem.

In the example of last article, the function

$$3x^2 - 5xy + 2y^2 - 15x + 6y - 8$$

contains some terms of the 2nd degree, others of the 1st, and the last term is of zero degree. Such a function is really a shortened form of the homogeneous function

$$3x^2 - 5xy + 2y^2 - 15ax + 6ay - 8a^2,$$

where a denotes the unit of length.

For, of course, every expression consisting of a sum of terms must really be homogeneous, *i e.* every term must be of the same *dimensions*, for it is impossible to add square feet, linear feet, and mere numbers together.

A function of any degree when rendered visibly homogeneous, as above, possesses certain important properties, of which we will now prove one of the most important, called Euler's theorem.

Euler's theorem is this:

If u is a homogeneous function of the nth degree in x, y, and z, then $x\frac{du}{dx}+y\frac{du}{dy}+z\frac{du}{dz}$ is identically equal to nu.

For u consists of a number of terms of the form $Kx^py^qz^r$ where $p+q+r=n$.

If we consider the result of operating on the above typical term, we find it becomes

$$x(pKx^{p-1}y^qz^r) + y(qKx^py^{q-1}z^r) + z(rKx^py^qz^{r-1}),$$

which reduces to $\qquad (p+q+r)Kx^py^qz^r$,

i.e. $\qquad n$ times $Kx^py^qz^r$.

Hence, if we operate on the whole of u, we shall obtain n times u, which proves the proposition.

Take as illustration

$$u=3x^2-5xy+2y^2-15xz+6yz-8z^2.$$

Here $$x\frac{du}{dx}=x(6x-5y-15z),$$

$$y\frac{du}{dy}=y(-5x+4y+6z),$$

and $$z\frac{du}{dz}=z(-15x+6y-16z),$$

whence, by addition,

$$x\frac{du}{dx}+y\frac{du}{dy}+z\frac{du}{dz}=2u,$$

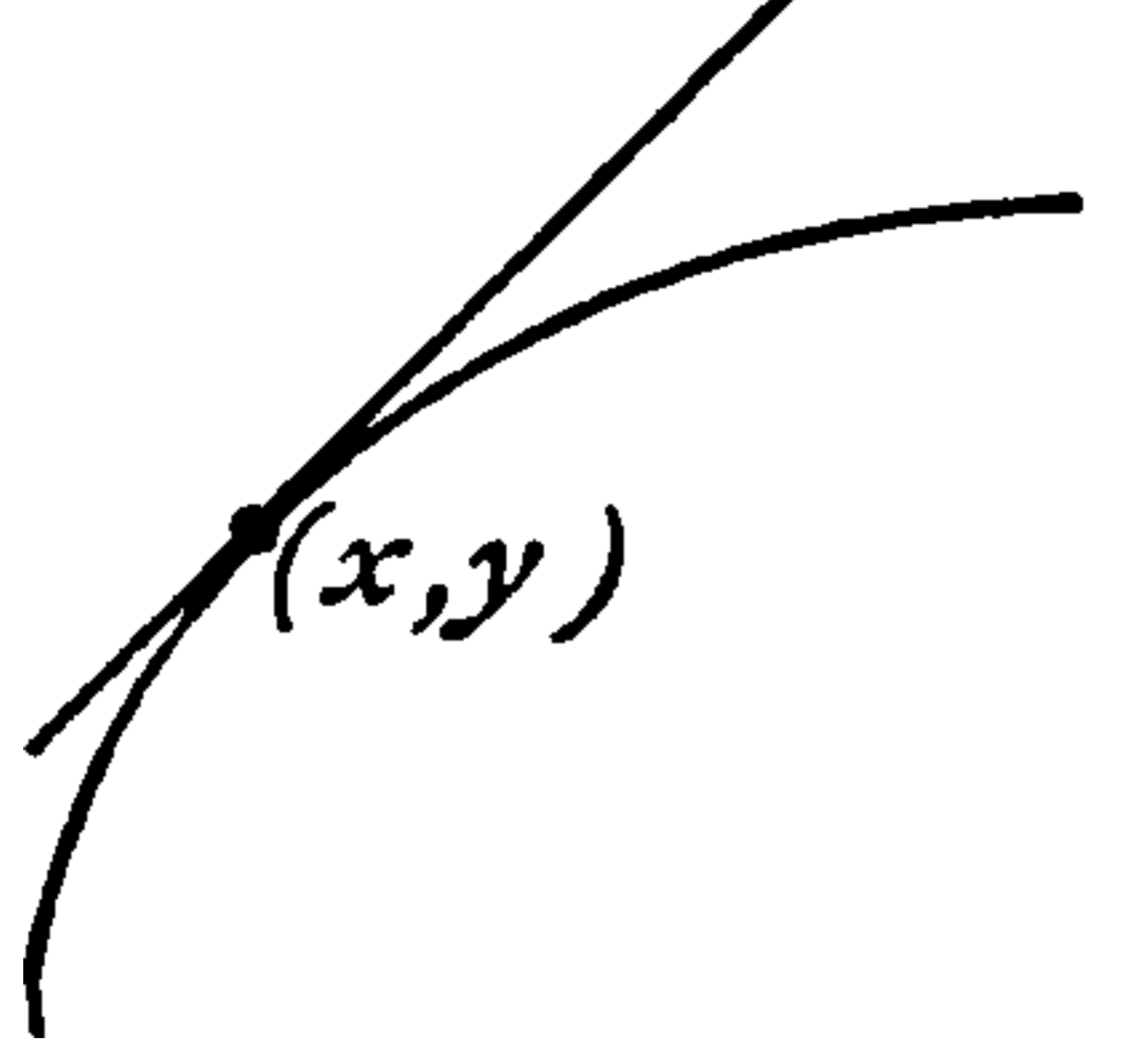
which agrees with the predicted result, since here $n=2$.

We shall make use of this theorem in the following investigation:

185. Equation of the tangent at any point of the curve u = 0.

We know that $$\frac{dy}{dx}=-\frac{\dfrac{du}{dx}}{\dfrac{du}{dy}}.$$

If, then, we take (x, y) for the point of contact, and (X, Y) as any random point on the tangent, we find for its equation



$$\frac{Y-y}{X-x}=-\frac{\dfrac{du}{dx}}{\dfrac{du}{dy}},$$

whence

$$X\frac{du}{dx}+Y\frac{du}{dy}=x\frac{du}{dx}+y\frac{du}{dy}.$$

This is one form of the equation to the tangent at (x, y), but it can be improved by help of Euler's theorem.

If u is made homogeneous by taking a as unit of length, we know that

$$x\frac{du}{dx}+y\frac{du}{dy}+a\frac{du}{da}=nu,$$

and this is equal to 0, since (x, y) is on the curve

Hence, finally, the equation of the tangent at (x, y) is

$$\mathbf{X\frac{du}{dx}+Y\frac{du}{dy}+a\frac{du}{da}=0.}$$

Example. Find the tangent at any point of the curve

$$3x^2-5xy+2y^2-15x+6y+20=0.$$

Making the equation homogeneous by taking a as the unit of length, we have

$$3x^2-5xy+2y^2-15ax+6ay+20a^2=0.$$

The equation of the tangent at any point (x, y) is

$$\mathrm{X}[6x-5y-15a]+\mathrm{Y}[-5x+4y+6a]+a[-15x+6y+40a]=0,$$

i.e. again putting $a=1$,

$$\mathrm{X}(6x-5y-15)+\mathrm{Y}(-5x+4y+6)=15x-6y-40.$$

If we want the tangent at any special point, *e.g.* (2, 1), we have only to insert the special values of x and y in the above equation. Thus the tangent at (2, 1) is

$$-8\mathrm{X}=-16, \text{ i.e. } \mathrm{X}=2.$$

N.B.—It is essential that the point (x, y) should really be on the curve. If not, the equation does not represent a tangent. The student should see that the point (2, 1) does satisfy the required condition.

186. Equation of Normal.

Since the normal is perpendicular to the tangent, it follows that if (X, Y) denotes any point on the normal, it satisfies the gradient equation

$$\frac{\mathrm{Y}-y}{\mathrm{X}-x}=\frac{\dfrac{du}{dy}}{\dfrac{du}{dx}},$$

or, as it is often written, for symmetry,

$$\frac{\mathrm{X}-x}{\left(\dfrac{du}{dx}\right)}=\frac{\mathrm{Y}-y}{\left(\dfrac{du}{dy}\right)}.$$

187. Polar of a Point.

The equation of a tangent at any point (x, y) of the curve $u=0$ is

$$\mathrm{X}\frac{du}{dx}+\mathrm{Y}\frac{du}{dy}+a\frac{du}{da}=0,$$

if a is the unit of length.

If we fix on any special point (X, Y) on the tangent, the above equation is satisfied by the coordinates (x, y) of the point of contact.

If more than one tangent can be drawn through (X, Y), the equation is satisfied by the coordinates (x, y) of every point of contact; *i.e.* the equation is the equation of some curve which passes through the points of contact of all tangents drawn from (X, Y). This curve is called the polar of the point (X, Y) with regard to the curve $u=0$. It is easy to see that its degree is in general $=n-1$ if $u=0$ is of the n^{th} degree, so that if $u=0$ is a conic section, the polar is a straight line; and if $u=0$ is of the 3^{rd} degree, the polar is a conic section.

188. Condition that an equation of the second degree shall represent straight lines (real or imaginary).

Let $u=0$ be the equation of the locus.

Its centre is given by $\frac{du}{dx}=0$, and $\frac{du}{dy}=0$.

If the centre is on the locus, the locus must be either a single point, or a pair of straight lines intersecting in this point; *i.e.* in either case, the locus must be a pair of real or imaginary straight lines passing through the point.

Now $u=0$ is, by Euler's theorem, equivalent to

$$x\frac{du}{dx}+y\frac{du}{dy}+a\frac{du}{da}=0.$$

The condition that the centre shall lie on the curve is obtained by substituting its coordinates in the equation. These coordinates make $\frac{du}{dx}=0$ and $\frac{du}{dy}=0$ (Art. 183).

Hence, if the centre is on the locus, its coordinates must also make $\frac{du}{da}=0$.

The condition, therefore, is that

$$\frac{du}{dx}=0, \quad \frac{du}{dy}=0, \quad \frac{du}{da}=0$$

shall be simultaneously true.

If the equation is

$$ax^2+2hxy+by^2+2gx+2fy+c=0,$$

the simultaneous equations are, on dividing by 2,

$$ax+hy+g=0,$$

$$hx+by+f=0,$$

$$gx+fy+c=0.$$

EXAMPLES.

1. Find the tangent and normal at any point of the following curves:

(1) $x^2+y^2=a^2$; (2) $y^2=4ax$;

(3) $Ax^2+By^2=a^2$; (4) $xy=a^2$.

2. Find the tangent at any point of the circle

$$x^2+y^2+Ax+By+C=0.$$

[Insert a as unit of length.]

3. Show that the curve $x^3+y^3=a^3$ cuts the axes at right angles. Find the polar of (a, a) with regard to this curve. Draw the part of the curve situated in the positive quadrant.

4. Find the polar of (h, k) with regard to the curve

$$xy^2-2ay^2+a^2x=0,$$

and, in particular, find the polar of the origin. Find where this polar cuts the curve, and draw the tangents from the origin to the curve.

5. Show that the equation $x^2-4xy+3y^2+2ax-2ay=0$ represents straight lines, and draw them.

6. Show that the equation $x^2-xy+y^2-2x-2y+4=0$ represents a pair of imaginary straight lines, intersecting in the point (2, 2).

189. Successive partial differential coefficients of a function $f(x, y)$ of two variables. Proof that

$$\frac{d}{dy}\left(\frac{df}{dx}\right)=\frac{d}{dx}\left(\frac{df}{dy}\right).$$

The partial differential coefficients of $f(x, y)$ with regard to x and y respectively, namely $\frac{df}{dx}$ and $\frac{df}{dy}$, are themselves, in general, functions of both x and y, and therefore have partial differential coefficients.

These will be $\qquad \frac{d}{dx}\left(\frac{df}{dx}\right)$ and $\frac{d}{dy}\left(\frac{df}{dx}\right)$;

and $\qquad \frac{d}{dx}\left(\frac{df}{dy}\right)$ and $\frac{d}{dy}\left(\frac{df}{dy}\right)$.

The first and last of these are called $\frac{d^2f}{dx^2}$ and $\frac{d^2f}{dy^2}$ respectively. The second and third are equal to each other, as we shall now show, and are called $\frac{d^2f}{dxdy}$ or $\frac{d^2f}{dydx}$ as we please.

Imagine the function to be expanded in powers of x and y, so that it consists of a series of terms of the form $\mathrm{K}x^py^q$.

Then the corresponding terms of $\frac{df}{dx}$ and $\frac{df}{dy}$ respectively will be $p\mathrm{K}x^{p-1}y^q$ and $q\mathrm{K}x^py^{q-1}$.

Hence the corresponding terms of $\frac{d}{dy}\left(\frac{df}{dx}\right)$ and $\frac{d}{dx}\left(\frac{df}{dy}\right)$ will each be $pq\mathrm{K}x^{p-1}y^{q-1}$, which proves the proposition.

This is easily extended to any number of differentiations, and any number of variables, the result being that if a function of two or more variables be differentiated a number of times with regard to one variable, and a number of times with regard to another variable, the order of performing the work is immaterial.

190. Homogeneous functions of two variables. Extension of Euler's first theorem.

If $f(x, y)$ is a homogeneous function, of the n^{th} degree, of two variables x and y, we have proved that

$$x\frac{df}{dx}+y\frac{df}{dy}=nu.$$

Now evidently $\frac{df}{dx}$ and $\frac{df}{dy}$ are two homogeneous functions of x and y, of the $n-1^{\text{th}}$ degree: hence

$$x\frac{d^2f}{dx^2}+y\frac{d^2f}{dxdy}=(n-1)\frac{df}{dx},$$

and

$$x\frac{d^2f}{dxdy}+y\frac{d^2f}{dy^2}=(n-1)\frac{df}{dy}.$$

Hence, multiplying the first by x, and the second by y, and adding, we obtain

$$x^2\frac{d^2f}{dx^2}+2xy\frac{d^2f}{dxdy}+y^2\frac{d^2f}{dy^2}=(n-1)\left(x\frac{df}{dx}+y\frac{df}{dy}\right)=n(n-1)u.$$

191. Successive total differential coefficients of a function of two variables x and y, which are themselves functions of a third variable t.

Let $u=f(x, y)$, then

$$\frac{du}{dt}=\left(\frac{df}{dx}\right)\frac{dx}{dt}+\left(\frac{df}{dy}\right)\frac{dy}{dt}.$$

Denote, for brevity,

$$\frac{dx}{dt},\ \frac{dy}{dt},\ \text{by}\ x_1,\ y_1;$$

and similarly

$$\frac{d^2x}{dt^2},\ \frac{d^2y}{dt^2},\ \text{by}\ x_2,\ y_2;$$

and so on. Then

$$\frac{du}{dt}=x_1\frac{df}{dx}+y_1\frac{df}{dy}.\ \dots\dots\dots\dots(1)$$

This may be written in the form

$$\frac{d}{dt}f(x, y)=\left(x_1\frac{d}{dx}+y_1\frac{d}{dy}\right)f(x, y),$$

indicating that if we wish to find the differential coefficient of a function of x and y with regard to t, the operation indicated by $\frac{d}{dt}$ must be replaced by the operation indicated by

$$x_1\frac{d}{dx}+y_1\frac{d}{dy}.$$

These expressions themselves are called operators, indicating certain definite operations to be performed.

There is nothing new in this, except the systematic way of considering and expressing the work which we know has to be performed when we differentiate such a function.

Now let us find $\frac{d^2u}{dt^2}$, *i.e.* $\frac{d}{dt}\left(x_1\frac{df}{dx}+y_1\frac{df}{dy}\right)$.

Evidently this equals [see Art. 24]

$$\left\{x_1\frac{d}{dt}\left(\frac{df}{dx}\right)+\frac{dx_1}{dt}\cdot\frac{df}{dx}\right\}+\left\{y_1\frac{d}{dt}\left(\frac{df}{dy}\right)+\frac{dy_1}{dt}\cdot\frac{df}{dy}\right\}.$$

Now $$\frac{d}{dt}\left(\frac{df}{dx}\right)=\left(x_1\frac{d}{dx}+y_1\frac{d}{dy}\right)\frac{df}{dx},$$

since $\frac{df}{dx}$ is a function of x and y. This equals

$$x_1\frac{d^2f}{dx^2}+y_1\frac{d^2f}{dxdy}.$$

Similarly $$\frac{d}{dt}\left(\frac{df}{dy}\right)=\left(x_1\frac{d}{dx}+y_1\frac{d}{dy}\right)\left(\frac{df}{dy}\right)$$

$$=x_1\frac{d^2f}{dxdy}+y_1\frac{d^2f}{dy^2}.$$

Also $$\frac{dx_1}{dt}=x_2,\text{ and }\frac{dy_1}{dt}=y_2.$$

Hence, on substituting, and arranging the terms, we find

$$\frac{d^2u}{dt^2}=\left(x_1{}^2\frac{d^2f}{dx^2}+2x_1y_1\frac{d^2f}{dxdy}+y_1{}^2\frac{d^2f}{dy^2}\right)+\left(x_2\frac{df}{dx}+y_2\frac{df}{dy}\right)\ldots.(2)$$

Similarly $\frac{d^3u}{dt^3}$, ... could be formed, but they rapidly increase in complexity.

If we are dealing with the curve $u =$ constant, all these total differential coefficients are zero, and the above equations then give the connections between the partial differential coefficients and the values of $x_1, y_1, x_2, y_2, \ldots$.

EXAMPLES.

1. If
$$u = x^3 - 3x^2y + 4xy^2 - y^3,$$
find
$$\frac{d}{dx}\left(\frac{du}{dy}\right) \text{ and } \frac{d}{dy}\left(\frac{du}{dx}\right).$$

2. If $u = x^3y^4$, find $\dfrac{d^3u}{dx^2dy}$ in several ways, *i.e.* by varying the order of the operations indicated, showing that they all lead to the same result.

3. In the function of Example 1, test the formula
$$x^2\frac{d^2u}{dx^2} + 2xy\frac{d^2u}{dx\,dy} + y^2\frac{d^2u}{dy^2} = 6u$$
by going through the operations.

4. Show that $x^2\dfrac{d^2u}{dx^2} + 2xy\dfrac{d^2u}{dx\,dy} + y^2\dfrac{d^2u}{dy^2}$ is not the same as
$$\left(x\frac{d}{dx} + y\frac{d}{dy}\right)^2 u.$$
Find the values of the two quantities in the case of the function of Example 1.

5. Show that $h^2\dfrac{d^2u}{dx^2} + 2hk\dfrac{d^2u}{dx\,dy} + k^2\dfrac{d^2u}{dy^2}$ *is* the same as $\left(h\dfrac{d}{dx} + k\dfrac{d}{dy}\right)^2 u$, if h and k are constants.

6. Find what the formula of Art. 191 for $\dfrac{d^2u}{dt^2}$ becomes when $t \equiv x$, *i.e.* when y is a given function of x.

7. Find what the same formula reduces to when $u \equiv y$.

8. Find what the same formula reduces to when $\dfrac{dx}{dt}$ and $\dfrac{dy}{dt}$ are constant. Take, for example, $x - a = ht$, $y - b = kt$, and compare Ex. 5 above.

9. If $u =$ constant, find the value of $\dfrac{d^2y}{dx^2}$ in terms of the successive partial differential coefficients:

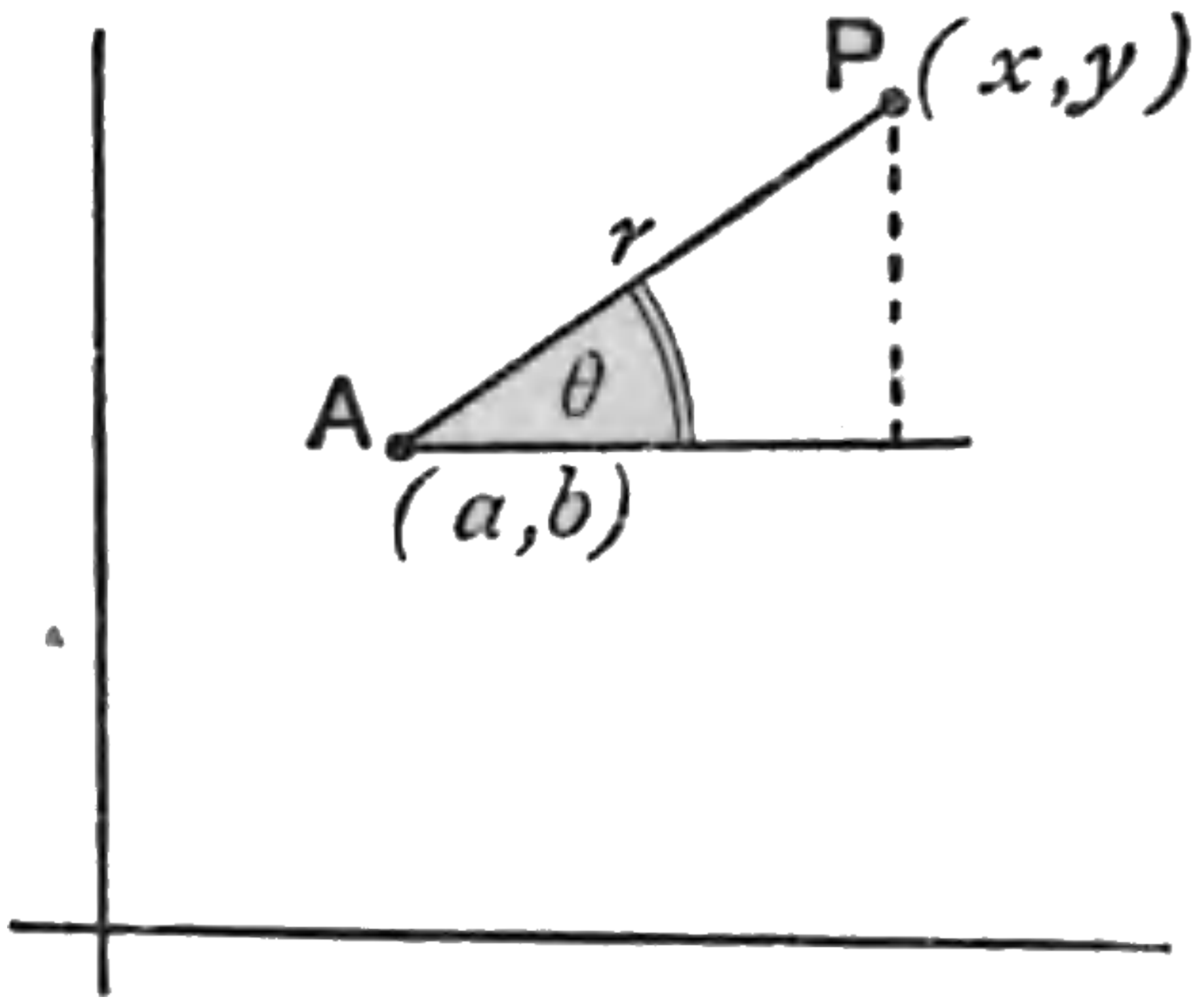
(1) by means of the formula obtained above (Ex. 6);

(2) by direct differentiation of the equation $\dfrac{dy}{dx} = -\left(\dfrac{du}{dx}\right) \div \left(\dfrac{du}{dy}\right)$.

192. Connection between rectangular coordinates and polar coordinates.

The position of a variable point in a plane may be given not only by means of its rectangular coordinates x and y, but also by means of what are called polar coordinates, viz. the distance of the point from the origin, or some other fixed point, and the angle that this line makes with the axis of x. These are called the polar coordinates of the variable point, and are usually denoted by the letters r and θ.

If r is measured from the fixed point (a, b), the connection between x, y and r, θ is



$$x - a = r\cos\theta,$$
$$y - b = r\sin\theta,$$

and the reverse relations are

$$r^2 = (x-a)^2 + (y-b)^2,$$
$$\tan\theta = \frac{y-b}{x-a}; \ \sin\theta = \frac{y-b}{r}; \ \cos\theta = \frac{x-a}{r}.$$

The differential relations between x, y and r, θ, if θ is expressed in radians, are

$$dx = \cos\theta\, dr - r\sin\theta\, d\theta,$$
$$dy = \sin\theta\, dr + r\cos\theta\, d\theta;$$

and

$$dr = \cos\theta\, dx + \sin\theta\, dy,$$
$$r\, d\theta = -\sin\theta\, dx + \cos\theta\, dy.$$

From these follow the partial differential coefficients:

$$\left(\frac{dx}{dr}\right) = \cos\theta, \quad \left(\frac{dx}{d\theta}\right) = -r\sin\theta,$$
$$\left(\frac{dy}{dr}\right) = \sin\theta, \quad \left(\frac{dy}{d\theta}\right) = r\cos\theta;$$

and

$$\left(\frac{dr}{dx}\right) = \cos\theta, \quad \left(\frac{dr}{dy}\right) = \sin\theta,$$
$$r\left(\frac{d\theta}{dx}\right) = -\sin\theta, \quad r\left(\frac{d\theta}{dy}\right) = \cos\theta.$$

It is important to realize exactly what the partial differential coefficients denote.

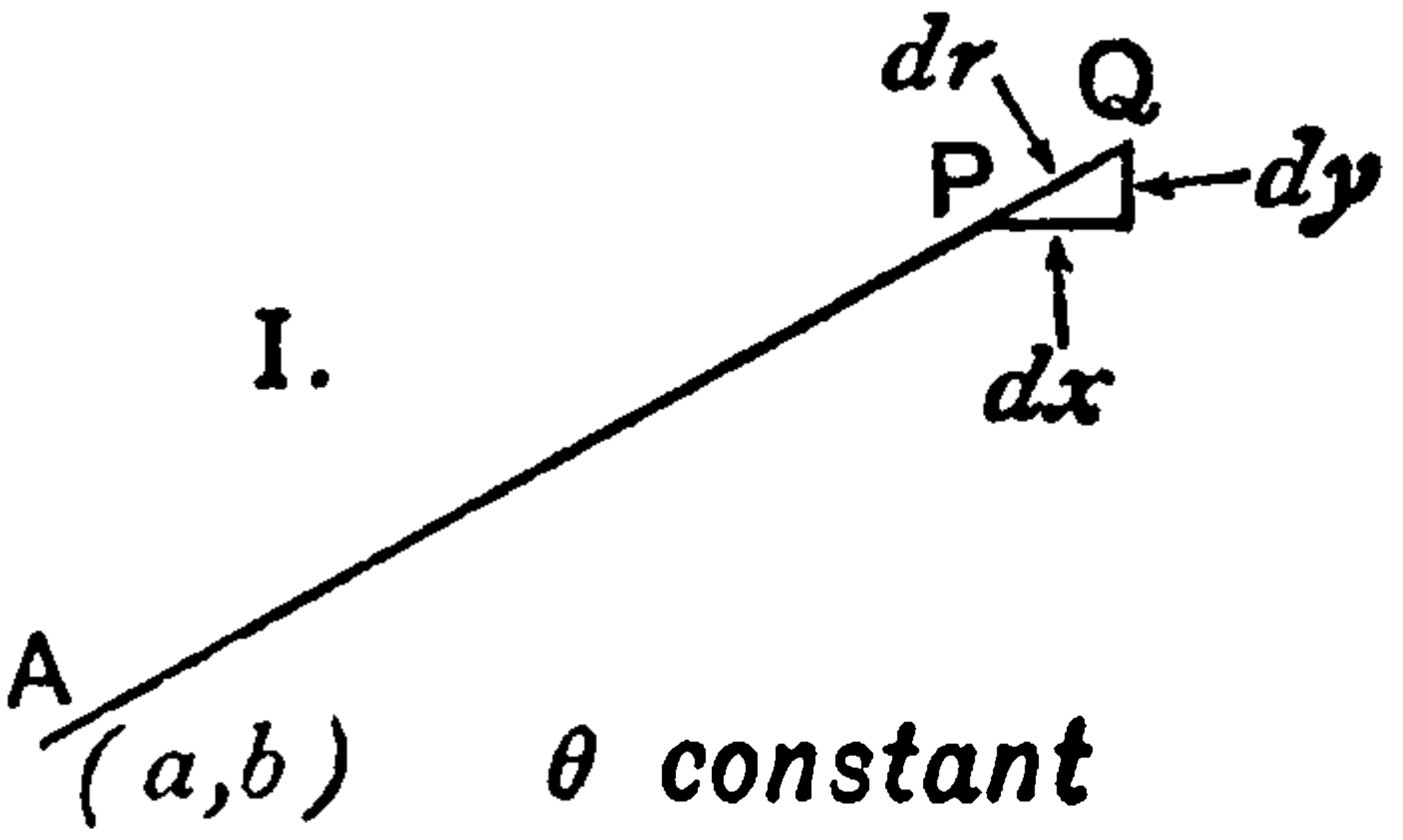
I. $\left(\dfrac{dx}{dr}\right)$ and $\left(\dfrac{dy}{dr}\right)$ denote the rates of change of x and y per unit change in r when x, y and r vary, but not θ.

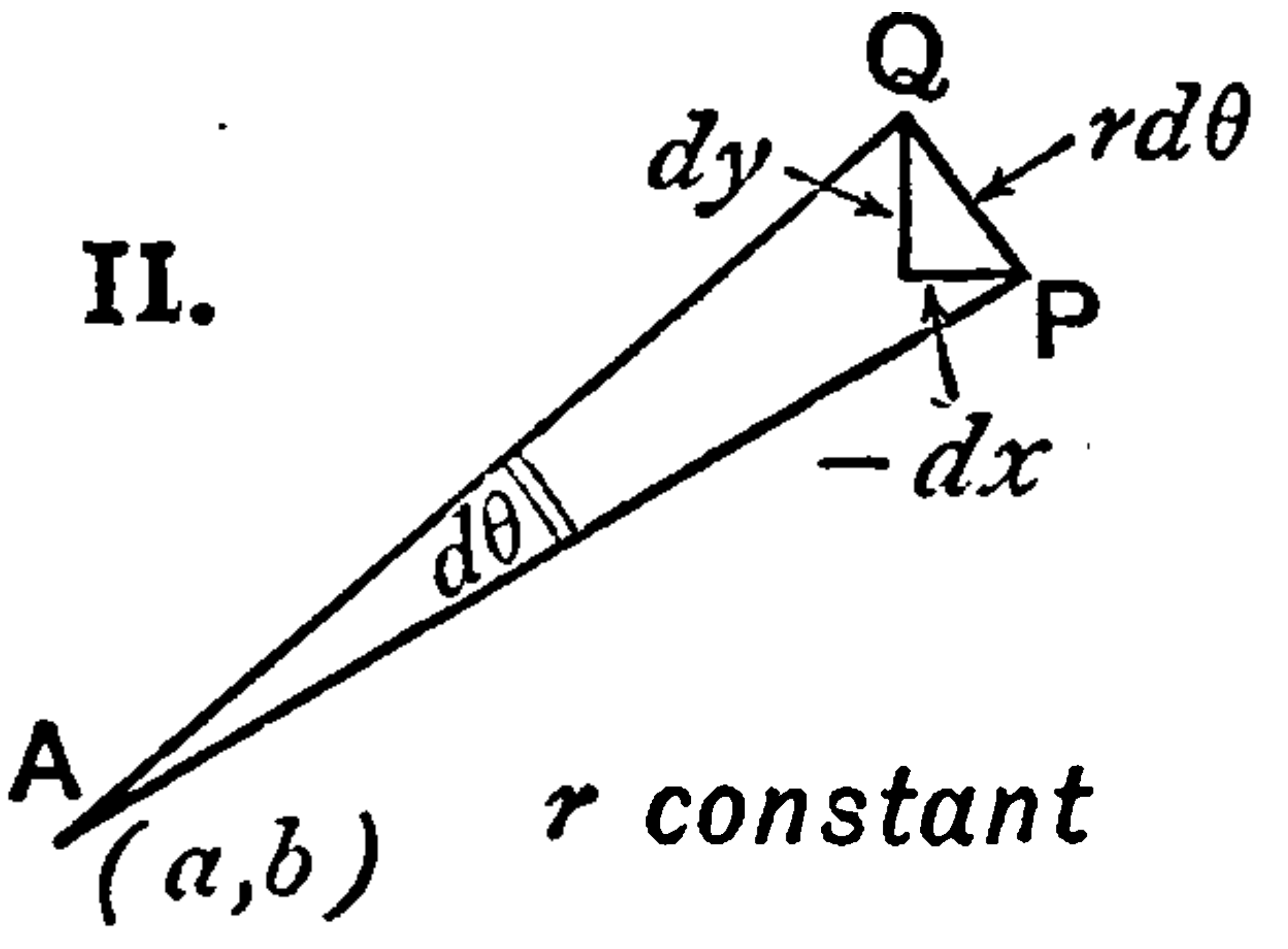
II. $\left(\dfrac{dx}{d\theta}\right)$ and $\left(\dfrac{dy}{d\theta}\right)$ denote the rates of change of x and y per unit change in θ when x, y and θ vary, but not r.

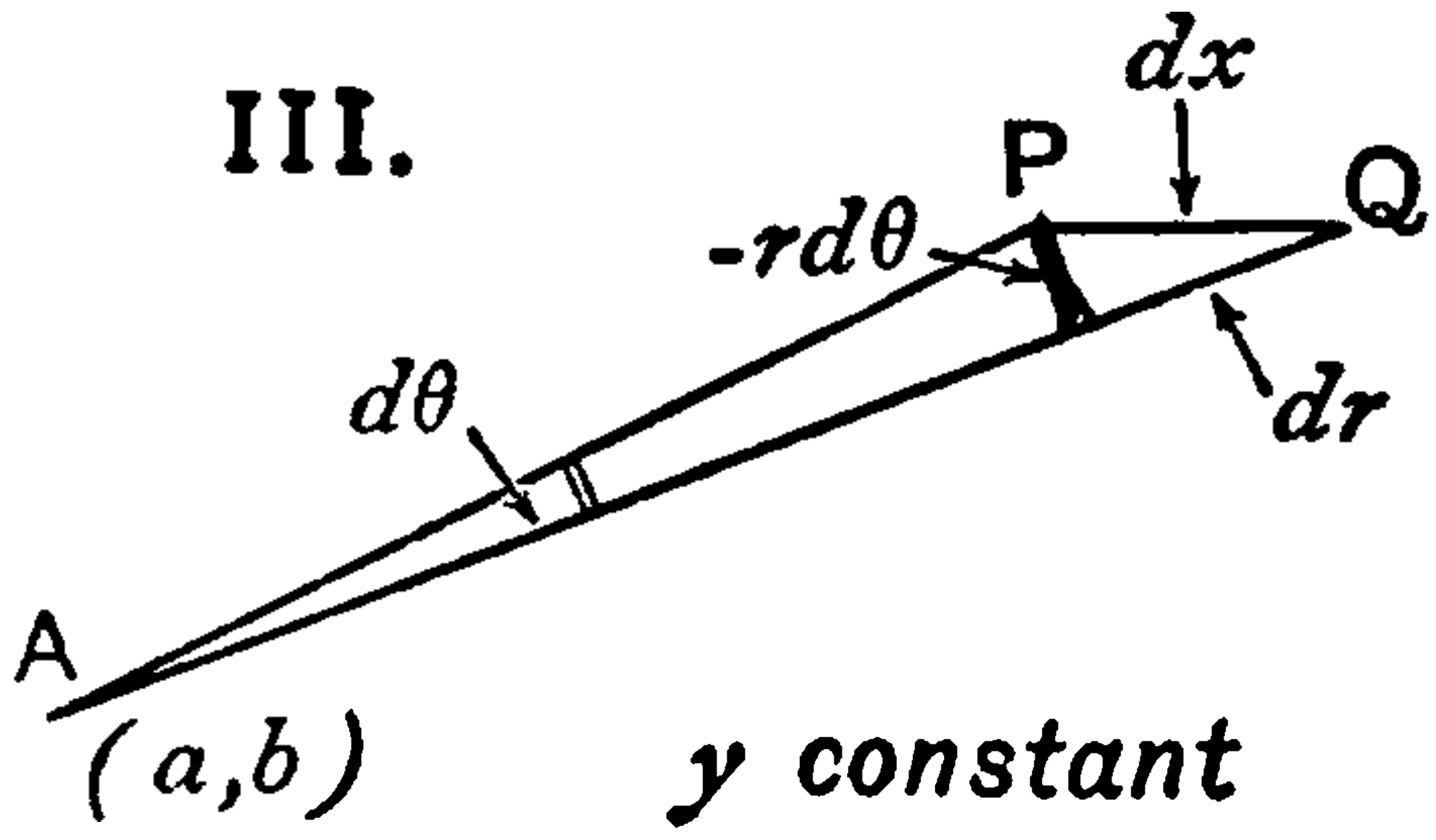
III. $\left(\dfrac{dr}{dx}\right)$ and $\left(\dfrac{d\theta}{dx}\right)$ are formed on the supposition that r, θ and x vary, but y does not.

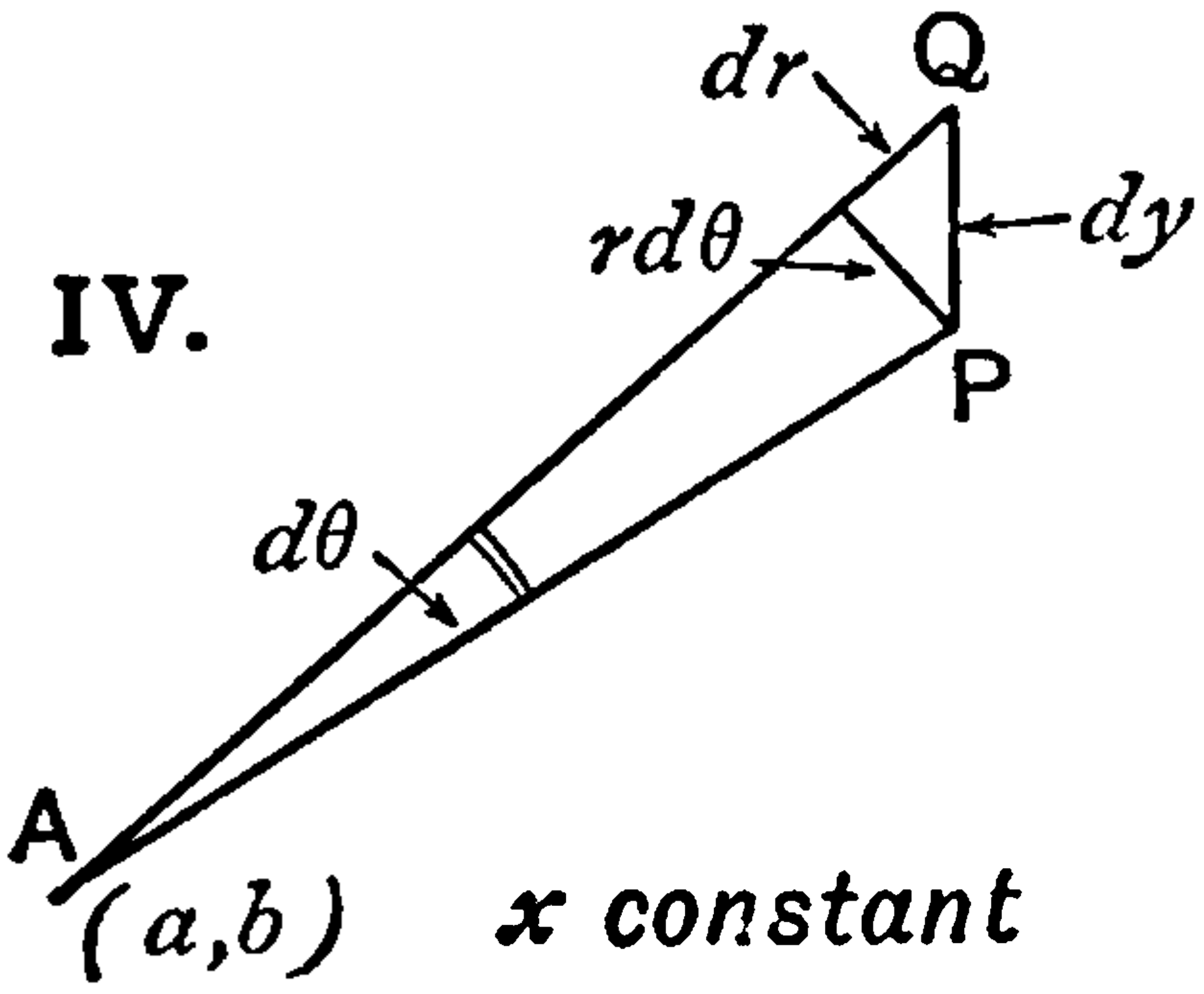
IV. $\left(\dfrac{dr}{dy}\right)$, $\left(\dfrac{d\theta}{dy}\right)$ assume r, θ, and y to vary, but not x.

These changes are shown in the following diagrams, where the point is supposed to move from P to Q, where PQ is infinitely small, but the direction of movement is in each case such that *one* of the quantities x, y, r, θ remains unaltered.









193. Application to functions.

If u is a function of x and y, it can also be expressed in erms of r and θ.

Now $$du = \left(\frac{du}{dx}\right)dx + \left(\frac{du}{dy}\right)dy.$$

Hence, if we treat u as a function of r and not of θ, *i.e.* if we keep θ constant, we find

$$\left(\frac{du}{dr}\right) = \left(\frac{du}{dx}\right)\cos\theta + \left(\frac{du}{dy}\right)\sin\theta. \quad\ldots\ldots\ldots\ldots\ldots(1)$$

Similarly, if we treat θ as the variable, r remaining constant, we find

$$\left(\frac{du}{d\theta}\right) = -\left(\frac{du}{dx}\right)r\sin\theta + \left(\frac{du}{dy}\right)r\cos\theta. \quad\ldots\ldots\ldots\ldots(2)$$

Again, since u is expressible as a function of r and θ,

$$du = \left(\frac{du}{dr}\right)dr + \left(\frac{du}{d\theta}\right)d\theta;$$

$$\therefore \quad \left(\frac{du}{dx}\right) = \left(\frac{du}{dr}\right)\cos\theta - \left(\frac{du}{d\theta}\right)\frac{1}{r}\sin\theta, \quad\ldots\ldots\ldots\ldots(3)$$

and $$\left(\frac{du}{dy}\right) = \left(\frac{du}{dr}\right)\sin\theta + \left(\frac{du}{d\theta}\right)\frac{1}{r}\cos\theta. \quad\ldots\ldots\ldots\ldots(4)$$

From these relations the second partial differential coefficients can be obtained. Thus, for example, from (1),

$$\begin{aligned}\left(\frac{d^2u}{dr^2}\right) &= \cos\theta\left\{\left(\frac{d^2u}{dx^2}\right)\cos\theta + \left(\frac{d^2u}{dxdy}\right)\sin\theta\right\} \\ &\qquad + \sin\theta\left\{\left(\frac{d^2u}{dxdy}\right)\cos\theta + \left(\frac{d^2u}{dy^2}\right)\sin\theta\right\} \\ &= \left(\frac{d^2u}{dx^2}\right)\cos^2\theta + 2\left(\frac{d^2u}{dxdy}\right)\cos\theta\sin\theta + \left(\frac{d^2u}{dy^2}\right)\sin^2\theta. \ldots(5)\end{aligned}$$

This relation is of importance in connection with the maximum and minimum values of a function of two variables, as we will proceed to show.

194. *To find the maximum and minimum values of a given function of two variables.*

Let u denote the function of two variables x and y, it is required to find for what values of x and y the value of u may be a maximum or minimum.

We can obtain a geometrical picture of the function by supposing x and y to be the rectangular coordinates of points in a horizontal plane, and the values of u to be denoted by the lengths of vertical lines drawn from those points, so that the tops of these vertical lines lie on an undulating surface like a tract of hilly country. At the top of a hill, u will be a maximum, and at the bottom of a valley u will be a minimum. To fix the ideas we will suppose the axis of x is drawn in the east and west direction (positive towards the east), and the axis of y in the north and south direction, positive towards the north.

Evidently in general the surface is horizontal at a point where u is a maximum or a minimum.

Hence $\left(\frac{du}{dx}\right)$ and $\left(\frac{du}{dy}\right)$ must equal zero at such a point. This is the first condition to be satisfied.

We must therefore solve simultaneously the equations

$$\left(\frac{du}{dx}\right)=0,$$

$$\left(\frac{du}{dy}\right)=0$$

to find the values of x and y for which the surface is horizontal.

Let $x=a$, $y=b$ be a solution of these equations.

If these values of x and y make u a maximum, it is evident (from our investigation of the conditions for a maximum value of a function of one variable) that $\left(\frac{d^2u}{dx^2}\right)$ and $\left(\frac{d^2u}{dy^2}\right)$ must be negative, as the conditions ensure that an east or west movement from the point, or a north or south movement from the point, shall be downhill.

This, however, is not sufficient, for if we are really at the top of a hill, any movement in *any direction* must begin to take us downhill. To investigate the necessary analytical condition for this to happen, take $x-a=r\cos\theta$, and $y-b=r\sin\theta$, and consider the movement through a short distance along a direction making any angle θ with the x-axis. If we are really at the top of a hill we must have $\left(\frac{du}{dr}\right)=0$ and $\left(\frac{d^2u}{dr^2}\right)$ negative, whatever θ may be.

Now $$\left(\frac{du}{dr}\right) = \left(\frac{du}{dx}\right)\cos\theta + \left(\frac{du}{dy}\right)\sin\theta,$$

so that the first condition, viz. $\left(\frac{du}{dr}\right) = 0$, is satisfied whatever θ may be, since, by supposition, both $\left(\frac{du}{dx}\right)$ and $\left(\frac{du}{dy}\right) = 0$.

Again, $$\left(\frac{d^2u}{dr^2}\right) = \left(\frac{d^2u}{dx^2}\right)\cos^2\theta + 2\left(\frac{d^2u}{dxdy}\right)\cos\theta\sin\theta + \left(\frac{d^2u}{dy^2}\right)\sin^2\theta$$
$$= \cos^2\theta\left\{\left(\frac{d^2u}{dx^2}\right) + 2\left(\frac{d^2u}{dxdy}\right)\tan\theta + \left(\frac{d^2u}{dy^2}\right)\tan^2\theta\right\}$$

Hence, if $\left(\frac{d^2u}{dr^2}\right)$ is to be negative for all values of θ, we must have $$\left(\frac{d^2u}{dx^2}\right) + 2\left(\frac{d^2u}{dxdy}\right)\tan\theta + \left(\frac{d^2u}{dy^2}\right)\tan^2\theta$$
negative for all values of θ.

The needful condition is that $\frac{d^2u}{dx^2}$ and $\frac{d^2u}{dy^2}$ shall be negative, and that $\left(\frac{d^2u}{dx^2}\right)\left(\frac{d^2u}{dy^2}\right) - \left(\frac{d^2u}{dxdy}\right)^2$ shall be positive.

To prove this, denote $\left(\frac{d^2u}{dx^2}\right)$, $\left(\frac{d^2u}{dxdy}\right)$, and $\left(\frac{d^2u}{dy^2}\right)$ by A, B, and C, and denote $\tan\theta$ by t.

We have to find the condition that
$$A + 2Bt + Ct^2$$
shall be negative for all values of t.

The expression is equal to
$$\frac{1}{C}\{(Ct)^2 + 2B(Ct) + AC\}$$
$$= \frac{1}{C}\{(Ct + B)^2 + AC - B^2\}.$$

This can only be negative for all values of θ if C is negative and $AC - B^2$ is positive. Q.E.D.

Cor. These conditions, of course, involve the condition that A shall be negative as well as C.

Similarly, the condition that u shall be a minimum for values of x and y that make $\frac{du}{dx}=0$ and $\frac{du}{dy}=0$ is that

$$\text{A and C shall be positive,}$$

and $AC-B^2$ positive. These conditions ensure that $\left(\frac{d^2u}{dr^2}\right)$ shall be *positive* for all values of θ

We see, therefore, that when $\left(\frac{du}{dx}\right)=0$ and $\left(\frac{du}{dy}\right)=0$ and

$$\left(\frac{d^2u}{dx^2}\right)\cdot\left(\frac{d^2u}{dy^2}\right)>\left(\frac{d^2u}{dxdy}\right)^2,$$

u will necessarily be either a maximum or a minimum, and, further, that u will be a maximum if $\left(\frac{d^2u}{dx^2}\right)$ and $\left(\frac{d^2u}{dy^2}\right)$ are negative, and a minimum if $\left(\frac{d^2u}{dx^2}\right)$ and $\left(\frac{d^2u}{dy^2}\right)$ are positive.

If $\left(\frac{d^2u}{dx^2}\right)\cdot\left(\frac{d^2u}{dy^2}\right)-\left(\frac{d^2u}{dxdy}\right)^2$ is negative, we shall have $\left(\frac{d^2u}{dr^2}\right)$ positive for some values of θ and negative for others, with two intermediate directions for which $\left(\frac{d^2u}{dr^2}\right)$ will be zero, viz. when

$$C\tan\theta=-B\pm\sqrt{B^2-AC}.$$

Such points on a hilly surface occur on a neck between two hills, or between two valleys.

195. Extension of Taylor's Theorem to a function of two variables.

Let $u=f(x, y)$ denote the function. It is required to find the connection between $f(a+h, b+k)$ and $f(a, b)$ when a, b are any pair of values of x, y, and $a+h$, $b+k$ are any other pair.

Let $x - a = r\cos\theta,\ \ y - b = r\sin\theta,$
so that when $x = a + h$, $y = b + k$, we have $r\cos\theta = h$, $r\sin\theta = k$.

Then $u = f(a + r\cos\theta,\ b + r\sin\theta)$, which is a function of r, if we take $\theta = \text{constant}$, *i.e.* if h, k are in a given ratio. (With this understanding, we may dispense with the brackets round $\frac{du}{dr}, \frac{d^2u}{dr^2}, \dots$ in the following work).

Hence, by Maclaurin's theorem,

$$u = u_0 + r\frac{du_0}{dr} + \frac{r^2}{1.2}\frac{d^2u_0}{dr^2} + \dots,$$

when $u_0, \frac{du_0}{dr}, \dots$, denote the values of $u, \frac{du}{dr}, \dots$, when $r = 0$, *i.e.* when $x = a$, $y = b$.

Now we have seen that $\frac{du}{dr} = \cos\theta\frac{du}{dx}^* + \sin\theta\frac{du}{dy}^*;$

$$\therefore\ \ r\frac{du}{dr} = h\frac{du}{dx} + k\frac{du}{dy}.$$

Also $$\frac{d^2u}{dr^2} = \cos^2\theta\frac{d^2u}{dx^2} + 2\cos\theta\sin\theta\frac{d^2u}{dxdy} + \sin^2\theta\frac{d^2u}{dy^2};$$

$$\therefore\ \ r^2\frac{d^2u}{dr^2} = h^2\frac{d^2u}{dx^2} + 2hk\frac{d^2u}{dxdy} + k^2\frac{d^2u}{dy^2},$$

which may be symbolically expressed as

$$r^2\frac{d^2u}{dr^2} = \left(h\frac{d}{dx} + k\frac{d}{dy}\right)^2 u.$$

Similarly $$r^3\frac{d^3u}{dr^3} = \left(h\frac{d}{dx} + k\frac{d}{dy}\right)^3 u.$$

....

Hence

$$u = f(a + h,\ b + k) = u_0 + \left(h\frac{d}{dx} + k\frac{d}{dy}\right)u_0 + \frac{1}{\underline{|2}}\left(h\frac{d}{dx} + k\frac{d}{dy}\right)^2 u_0 + \dots,$$

---

* These are *obviously* partial differential coefficients, so the brackets may be omitted.

where on the right-hand side $u_0, \frac{du_0}{dx}, \frac{du_0}{dy}$, etc., are the values of $f(x, y), \frac{df}{dx}, \frac{df}{dy}$, etc., when $x=a$, $y=b$.

This may be more conveniently represented in the form

$$f(x+h,\ y+k)=f(x,\ y)+\left(h\frac{d}{dx}+k\frac{d}{dy}\right)f(x,\ y)+\dots,$$

and may be briefly designated symbolically as

$$f(x+h,\ y+k)=\left\{1+\left(h\frac{d}{dx}+k\frac{d}{dy}\right)+\frac{1}{\underline{|2}}\left(h\frac{d}{dx}+k\frac{d}{dy}\right)^2+\dots\right\}f(x,\ y)$$

$$=e^{h\frac{d}{dx}+k\frac{d}{dy}}f(x,\ y).$$

This formula is the extension of Taylor's theorem to a function of two variables. It may be similarly extended to a function of any number of variables.

EXAMPLES.

Discuss the maximum and minimum values (if any) of

1. $4x^2+12xy+10y^2-4x-4y-2$.

2. $x^2-4xy+3y^2+2x-2y+6$.

3. x^3+y^3-6xy.

4. $6x^3y^2-x^4y^2-x^3y^3$.

5. $x^4+y^4-4x^2+4xy-4y^2$.

6. Show that the maximum and minimum values of $\left(\frac{d^2u}{dr^2}\right)$, *i.e.* of $A\cos^2\theta+2B\cos\theta\sin\theta+C\sin^2\theta$, are $\frac{1}{2}\{(A+C)\pm\sqrt{(A-C)^2+4B^2}\}$, and that they occur when $\tan 2\theta=\frac{2B}{A-C}$.

7. Show that these maximum and minimum values have the same sign when $AC>B^2$, and that they are respectively $A+C$ and 0 when $AC=B^2$.

8. Deduce by means of Ex. 6 that the axes of the ellipse obtained by giving any constant value to the expression in Ex. 1 lie in the directions given by $\tan 2\theta=-2$.

9. Show that in Ex. 2 the value of $\left(\frac{d^2u}{dr^2}\right)$ is zero when

$$\cos^2\theta-4\cos\theta\sin\theta+3\sin^2\theta=0.$$

Explain this geometrically in connection with the point where $\left(\frac{du}{dx}\right)$ and $\left(\frac{du}{dy}\right)$ are zero.

10. If $u=\frac{2x+3y+6}{x^2+y^2+1}$, show that $\frac{du}{dx}=\frac{2-2ux}{x^2+y^2+1}$, and that

$$\frac{du}{dy}=\frac{3-2uy}{x^2+y^2+1}.$$

Hence find the maximum and minimum values of u.

11. In the last question, show that when u is a maximum or minimum, $\left(\frac{d^2u}{dr^2}\right)=-\frac{2u}{x^2+y^2+1}$, and is therefore the same for all values of θ.

12. By using Taylor's theorem for the expansion of $f(x+h,\ y+k)$ in powers of h and k, deduce the conditions for a maximum or minimum value of $f(x,\ y)$.

13. Expand $f(x+h,\ y+k)$ in powers of h and k by Taylor's theorem when $f(x,\ y)$ is any one of the functions considered in Examples 1 to 5.

[Operate successively by $h\frac{d}{dx}+k\frac{d}{dy}$, $\frac{1}{2}\left(h\frac{d}{dx}+k\frac{d}{dy}\right)$, $\frac{1}{3}\left(h\frac{d}{dx}+k\frac{d}{dy}\right)$ ...]

14. Expand the functions of Examples 1 and 2 in powers of x and y, and determine h, k so that the coefficients of the first powers of x and y disappear. [Use Taylor's theorem for the expansion, merely interchanging x, y with h, k.]

15. Plot on a diagram values of the function $u=x^3+y^3-6xy$, for integer values of x and y between ±5, using 1 inch as unit of length. Draw (approximately) on the diagram the curves corresponding to $u=0$, $u=27$, and $u=-8$, noting that in the last case the locus breaks up into a straight line and a point.

196. Contour lines and lines of flow.

If we consider a point anywhere on the side of a hill, there is always one direction in which motion is neither up nor down. If we walk round the hill in such direction that we keep at the same level, we traverse what is called a contour line or line of level.

In the case of the surface representing the function u, the level direction at any point is given by $\left(\frac{du}{dr}\right)=0$,

$$i.e.\ \left(\frac{du}{dx}\right)\cos\theta+\left(\frac{du}{dy}\right)\sin\theta=0,$$

$$i.e.\ \tan\theta=-\frac{\frac{du}{dx}}{\frac{du}{dy}}.$$

This, as we have before proved, is the direction of the tangent to the curve $u=$ constant.

In fact the curves $u=$ constant are the equations of the contour lines of the given surface. They can be drawn on the map. [For the cases when a contour line reduces to a point, or contains a double, or crossing, point, see Art. 194.]

The direction at right angles to the above value of θ is the direction of steepest slope, the direction in which water would begin to flow from the point down the surface.

This direction is given by

$$\tan\theta=\frac{\dfrac{du}{dy}}{\dfrac{du}{dx}}.$$

The maximum value of $\left(\dfrac{du}{dr}\right)$ is thus found to be

$$\sqrt{\left(\frac{du}{dx}\right)^2+\left(\frac{du}{dy}\right)^2}$$

with a + sign if we go uphill, and a – sign if we go downhill. [See Ex. 18, p. 109.]

If we travel continuously in the direction of steepest slope, we traverse what is called a line of flow. These lines and their traces on the map are at right angles to the contour lines $u=$ constant.

If $v=$ constant is the trace of a line of flow, its direction at any point is given by

$$\tan\theta=-\frac{\dfrac{dv}{dx}}{\dfrac{dv}{dy}}.$$

Hence the curves satisfy at every point the differential equation

$$\frac{\dfrac{dv}{dx}}{\dfrac{dv}{dy}}=-\frac{\dfrac{du}{dy}}{\dfrac{du}{dx}},$$

$$i.e.\ \frac{dv}{dx}\cdot\frac{du}{dx}+\frac{dv}{dy}\cdot\frac{du}{dy}=0.$$

EXAMPLES.

1. Prove that the parabolas $y^2=a^2-2ax$, and $y^2=b^2+2bx$ intersect at right angles at the points $(\frac{1}{2}(a-b),\ \pm\sqrt{ab})$.

2. Draw the parabolas for $a=1, 2, 3$ and $b=1, 2, 3$.

3. Prove that the curves $4x^2-8y^2=a^2$ and $x^2y=b^3$ cut each other at right angles. Draw them for the case $a=5$, $b=1$.

4. Prove that the curves $2x^2+y^2=a^2$ and $y^2=bx$ cut each other at right angles. Draw them for $a=1, 2, 3$; $b=1, 2, 3$.

5. Prove that if $u=f(\mathrm{A}x^2-\mathrm{B}y^2)$, and $v=\mathrm{F}(x^{\mathrm{B}}y^{\mathrm{A}})$, the curves $u=$ constant, and $v=$ constant, cut each other at right angles.

6. Prove that the confocal conics

$$\frac{x^2}{a^2}+\frac{y^2}{a^2-c^2}=1 \text{ and } \frac{x^2}{b^2}+\frac{y^2}{b^2-c^2}=1$$

cut each other orthogonally at the points

$$x=\pm\frac{ab}{c},\quad y=\pm\frac{1}{c}\sqrt{(a^2-c^2)(c^2-b^2)}.$$

Draw them for the case $a=5$, $b=3$, $c=4$.

7. Prove that the cycloid

$$x=a(\theta-\sin\theta),\quad y=a(1-\cos\theta)$$

cuts the cycloid

$$x=a(\theta-\sin\theta),\quad y=a(1+\cos\theta)$$

at right angles.

8. If

$$u=\frac{x^2+y^2+c^2}{x},\quad v=\frac{x^2+y^2-c^2}{y},$$

show that the curves $u=a$, $v=b$ cut orthogonally. Draw them for $a=c, 2c, 3c$; $b=c, 2c, 3c$.

9. Show that the feet of the normals drawn from any point (h, k) to the curve $u=0$ lie on the curve

$$(x-h)\frac{du}{dy}=(y-k)\frac{du}{dx}.$$

10. If $u=0$ is a conic, and (h, k) its centre, show that the equation given in No. 9 represents the axes of the conic. [This theorem follows at once if we refer the conic to its principal axes.]

11. Show that the axes of the conic

$$3x^2-5xy+2y^2-15x+6y+20=0$$

are given by the equation

$$5x^2+2xy-5y^2+378x-330y-765=0.$$

12. Show, by the method of Chap. XII., that the gradients of the tangent at a double point are given by the equation $\tan\theta=-\dfrac{\mathrm{A}+\mathrm{B}\tan\theta}{\mathrm{B}+\mathrm{C}\tan\theta}$ leading at once to the last equation of Art. 194.

CHAPTER XIV.

CURVE-TRACING. RECTANGULAR COORDINATES.

197. It is important to be able to draw a curve from its equation. For this purpose a certain amount of experience is necessary to enable the important characteristics of the curve, and its general shape, to be readily ascertained. We shall, in this chapter, consider curves given by rectangular coordinates, and, in the next chapter, those given by polar coordinates. In the former case, there is a distinction between those in which y is given as an explicit function of x (or x as an explicit function of y), such as

$$y=x^2(6-x),\ y=\sin x,\ y=e^x,\ldots$$

and those in which x and y are implicit functions of each other, such as $x^3+y^3-6xy=0,\ x^2+y^2-2ax=0,\ \ldots$

We already know how to plot a curve when y is an explicit function of x, viz. by making a tabular list of values of x, y and $\dfrac{dy}{dx}$, and plotting a series of points and gradients, and drawing the curve through those points at the proper gradients, paying particular attention to the points where y is a maximum or minimum, *i.e.* where the gradient is zero. So also if x is given as an explicit function of y, we must tabulate x, y, $\dfrac{dx}{dy}$ and plot similarly, giving the axis of y the post of honour instead of the axis of x. Or, again, it may be possible to express x and y as explicit functions of some third variable; *e.g.* in the case of the parabola $y^2=4ax$, we may write $x=at^2$, $y=2at$, where t is any variable. In such cases, we can easily also express $\dfrac{dy}{dx}$

in terms of the third variable, and then find as many points and gradients of the curve as we please by simply making a table of values of x, y and $\frac{dy}{dx}$ corresponding to different values of the variable on which they depend. Examples of equations allowing of this method of treatment will be found in the sequel. We shall say no more here about these particular forms of equations, but confine ourselves to the general case (of course including the above simple cases) in which the equation is in the form $f(x, y)=0$, *i.e.* in which x and y are, in general, implicit functions of each other.

198. Algebraic curves. Degree of a curve. Equations and curves may be classified into Algebraic and Transcendental; *e.g.* $y=2x-x^2$ is algebraic, and so are $y=\sqrt{(2x-x^2)}$ and $y+\sqrt{(2x-y)}=0$, while $y=\sin x$, $y=\log x$, are transcendental. We shall confine our attention solely to algebraic curves whose equations have been cleared of roots and fractions, *i.e.* in which every term is of the form Ax^ry^s, where r and s are positive integers or zero, and A is any numerical coefficient, as for example,

$$(1)\ x^2+y^2+3x-2y=0.$$

$$(2)\ x^3-xy^2+y^2-4=0.$$

$$(3)\ x^3-6y^2+y^3=0.$$

$$(4)\ x^2y^2=a^2(x^2-y^2).$$

$$(5)\ x^2(x-y)+a^2(x+y)=0.$$

The student can have these curves in his mind while reading the following articles, and should endeavour to trace them gradually as he reads.

The degree of the highest term or terms in the equation is called the degree of the equation and of the curve. Thus the first is of the 2nd degree; the fourth is of the 4th degree; and the others are of the 3rd degree.

199. Curve passing through the origin. If the equation contains no constant term, the origin is on the curve. In this case, the approximate shape of the curve near the origin can be discovered by neglecting the terms of high degree, and retaining only those of low degree, since, when x and y are small, the lower degree terms are more important than those of high degree (see Arts. 177, 178). And, in particular, if only the lowest degree terms are retained, we obtain the tangent or tangents to the curve at the origin.

If the lowest degree terms are of the first degree, there is one branch of the curve passing through the origin; if of the second degree, there are in general two branches passing through the origin, since on equating these terms to zero, we have in general two tangents. In this case the origin is called a *double point* on the curve. The two tangents thus obtained may be coincident: in such a case there are, in general, two branches passing through the origin and touching each other there; but in special cases the two branches may stop at the origin instead of going through it. The curve is then said to have a *cusp* at the origin. (Examples will be found below.) If the lowest degree terms cannot be broken up into real factors, the tangents are imaginary, and in this case the origin is an isolated point on the locus, separate from the rest of the curve. Such a point is called a *conjugate point.* If the equation giving the tangents at the origin is higher than the second degree, there may be several tangents, *i.e.* several branches of the curve passing through the origin. It is then called a *multiple point* on the curve. If all the tangents are imaginary, it is, of course, an isolated, or conjugate, point.

EXAMPLES.

1. Find the tangents at the origin in the case of the curves (1), (3), (4) and (5) above.

2. Draw the curve $x^3 = 6y^2$ in the neighbourhood of the origin.

3. Draw the curves $ay = x^2$, $a^2y = x^3$, $y^3 = ax^2$, $y^4 = a^2x^2$, between $x = \pm a$.

4. Show that the curve $y^2(3x+2y) = 4y^2 + 9x^2$ has a conjugate point at the origin.

5. Show, by the graphic method of Art. 178, that near the origin the shape of the curve $x^4 + y^4 - 2axy^2 - 3a^2x^2 = 0$ is approximately given by the equation $y^4 - 2axy^2 - 3a^2x^2 = 0$, *i.e.* by the parabolas $y^2 - 3ax = 0$, and $y^2 + ax = 0$. [Notice that in this case there are two *coincident* tangents at the origin, given by the equation $x^2 = 0$, and the curve has two branches, both touching the axis of y at the origin.]

200. Double and conjugate points in general. If we carefully study the system of contour lines $u=$ constant in connection with Arts. 194, 196, we see that a conjugate point occurs as part of the locus whenever u is a maximum or minimum, *i.e.* whenever the top of a hill, or the bottom of a valley, is on the level of the particular contour; and that a double point occurs when a neck between two hills or two valleys is on the level of the contour, which then, in general, crosses itself at this point. A triple or higher multiple point is far rarer and is not so easily pictured to the mind. All these points are classified as singular points.

We saw in last article how to discover and distinguish between conjugate and double points when they occur at the origin. The investigation in Art. 194 shows that at such points, in general, we must have $\frac{du}{dx}=0$ *and* $\frac{du}{dy}=0$; and, further, that the directions of the tangents at such a point are given by the equation

$$\left(\frac{d^2u}{dy^2}\right)\tan^2\theta+2\left(\frac{d^2u}{dxdy}\right)\tan\theta+\left(\frac{d^2u}{dx^2}\right)=0,$$

so that if the roots of this equation are real, it is a double point, which may be a cusp if the roots are equal; and, if they are imaginary, it is a conjugate point.

We have only mentioned this matter here because of its geometrical interest in connection with contour lines, but we shall not trouble the student with examples, as it is somewhat beyond the scope of the book.

We should note, however, that if, to assist us in drawing any curve $u=$constant, we draw the auxiliary curves $\frac{du}{dx}=0$ and $\frac{du}{dy}=0$, the gradient is zero where the curve $\frac{du}{dx}=0$ cuts the given curve, and infinite where $\frac{du}{dy}=0$ cuts the given curve, unless it happens that the two auxiliary curves intersect each other on the given curve, in which case the point of intersection is either a double point or a conjugate point on the given curve.

Of course if we are considering the whole *series* of curves $u =$ constant, the positions of these double or conjugate points are of great importance, as some of the series are bound to contain them. The centre of a hyperbola and the centre of an ellipse are, in this sense, examples of double and conjugate points respectively, the one being the crossing point of the asymptotes of a series of hyperbolas, and the other coinciding with the infinitely small ellipse belonging to a series of ellipses, the series in each case being obtained by giving different values to the constant in some second degree equation $u =$ constant. See Examples 5 and 6, p. 217, and draw several curves of each series, by equating the respective functions to different constants (such as 0, 2, 4, 6).

201. Simple cases of symmetry. If only even powers of y occur in an equation, the line $y = 0$ is a line of symmetry (as for example in the case of the parabola $y^2 = 4ax$), since for every value of x the values of y must be equal in magnitude and opposite in sign, so that for every point (x', y') above the line there must be a point $(x', -y')$ at an equal distance below; hence, if the upper part is folded-over round the axis, it will coincide with the lower part. The knowledge of this symmetry will be of great assistance in drawing the curve.

Similarly, if only even powers of x occur, the line $x = 0$ is a line of symmetry.

Thus, in (2) of Art. 198, $y = 0$ is a line of symmetry; and, in (4), both axes are lines of symmetry.

If all the terms in an equation are of even degree, as in the case of curve No 4 above, or if all the terms are of odd degree, as in the case of curve No. 5, the origin is what is called the centre of the curve, *i.e.* all chords passing through the origin are bisected there. To prove this, it is merely needful to see that if a point (x', y') is on the curve, the point $(-x', -y')$ is also on the curve, which is the case since the equation will be unaltered if $(-x', -y')$ is substituted for (x', y').

If the equation is unaltered when x and y are interchanged (*e.g.* $x^3 - 3axy + y^3 = b^3$), the curve is symmetrical about the line $x = y$. For if (x', y') is a point on the curve, (y', x') must also be on it, and these points will be found (by folding or otherwise) to be equidistant from the line $x=y$, and situate on a line perpendicular to it.

202. Curves which extend to infinity—Asymptotes.

An asymptote to a curve is a line which is infinitely close to the curve at points infinitely distant from the origin, *i.e.* at points whose abscissa and ordinate (or one of them) are infinitely great. Not only must the ratio of $y:x$ be the same for the line as for the curve, but the difference of the ordinates of points on the line and curve must be zero when the abscissa is infinite, and the difference of their abscissae must be zero when the ordinate is infinite.

All lines parallel to an asymptote at a finite distance from it will satisfy the condition that the ratio of $y:x$ will be the same as that of the curve, but they will not satisfy the difference condition. The asymptote is the line which satisfies both conditions.

The direction of this system of parallel lines is said to be the direction in which the curve extends to infinity.

In many cases the curve extends to infinity in more than one direction. The direction or directions of the infinite branches are found by neglecting all the terms in the equation except the highest, since, when x and y are great, their highest powers are more important than their lower powers. By equating the terms of highest degree to zero we obtain a homogeneous equation of the n^{th} degree representing n straight lines (of which an even number may be imaginary) through the origin stretching in the direction of the infinite branches of the curve. If all are imaginary, the curve does not extend to infinity in any direction, and is a closed curve. For example, in the hyperbola $b^2x^2-a^2y^2=a^2b^2$, the lines $b^2x^2-a^2y^2=0$, *i.e.* $bx \pm ay=0$, stretch towards the infinite parts of the curve, being in fact the asymptotes. In the ellipse $b^2x^2+a^2y^2=a^2b^2$ the corresponding lines are $b^2x^2+a^2y^2=0$, and are imaginary.

As a rule, the lines through the origin are not the asymptotes themselves, but are parallel to the asymptotes.

To find the asymptotes we have to make use of the next highest terms as well as the highest, and approximate by taking in succession the different factors of the highest degree as constant. The method of determining the value of the constant in each case has been already explained in Arts. 173, 174, and will now be illustrated by the following examples (taken from the curves of Art. 198.) The equations so obtained will be the equations of the asymptotes.

Examples.

(1) Let the eqnation of the curve be $x^3-6y^2+y^3=0$.

The terms of highest degree are x^3+y^3, therefore there is an asymptote in the direction $x+y=0$.

Writing the equation in the form $x+y=\frac{6y^2}{x^2-xy+y^2}$, and approximating to the right-hand side by the method of Art. 174, *i.e.* by putting $x=1$, $y=-1$ (equivalent to making the ratio $y:x=-1$), we find the value of the fraction is 2; $\therefore$ the asymptote is $x+y=2$.

(2) Let the equation of the curve be $x^2y^2=a^2(x^2-y^2)$.

The infinite directions are $y=0$ and $x=0$.

Considering first the asymptotes in the direction $y=0$, we find $y^2=\frac{a^2(x^2-y^2)}{x^2}=a^2$ when x is infinite (y being finite);

$$\therefore\ y=\pm a \text{ are a pair of parallel asymptotes.}$$

Again, $x^2=\frac{a^2(x^2-y^2)}{y^2}=-a^2$ when y is infinite.

These lines, $x^2+a^2=0$, are imaginary, hence the curve does not go to infinity in the direction $x=0$.

(3) In the curve $x^2(x-y)+a^2(x+y)=0$, we have

$$x^2=-a^2\frac{x+y}{x-y}=a^2 \text{ when } y \text{ is infinite,}$$

and
$$x-y=-a^2\frac{x+y}{x^2}=0 \text{ when } x \text{ and } y \text{ are infinite.}$$

$$\therefore\ x=\pm a,\ x-y=0, \text{ are all asymptotes.}$$

NOTE.—By putting the equation in the form $y=\frac{x^3+a^2x}{x^2-a^2}$ we can find as many points and gradients as we please, and so draw the curve. (See Art. 197.)

The tangent at the origin is $x+y=0$.

(4) The student can in the same way show that the asymptotes to the curve $x^3-xy^2+y^2-4=0$ are $x=1$ and $x\pm y=-\frac{1}{2}$.

203. Parallel asymptotes.

In some of the above cases there were a pair of parallel asymptotes, the finding of which was done in the same way as in the case of a single asymptote, with the peculiarity that the parallel asymptotes were sometimes imaginary.

The case of parallel asymptotes requires, however, further consideration, as may be shown by the following example.

Let the curve be

$$x^3+x^2y-xy^2-y^3+2xy+2y^2-3x+y=0,$$

i.e.

$$(x-y)(x+y)^2+2y(x+y)-3x+y=0.$$

There is one asymptote in the direction $x-y=0$, and a possibility of two asymptotes in the direction $x+y=0$.

The first presents no difficulty, and is easily seen to be $x-y+1=0$ by the method explained above.

To find the other two, divide by $x-y$, and approximate by the method of Arts. 174, 175, remembering that, though x and y are infinite where the asymptotes meet the curve, yet $x+y$ is finite.

The equation is

$$(x+y)^2+\frac{2y(x+y)}{x-y}-\frac{3x-y}{x-y}=0.$$

Now, in the first fraction, the numerator is of higher degree than the denominator, so that, in general, the fraction would be infinite when x and y are infinite. It happens, however, that the numerator contains the finite factor $x+y$, hence the limiting value of this fraction is $-(x+y)$, since the limiting value of $\frac{2y}{x-y}$ under the given conditions is -1 (Art. 174).

The limiting value of the second fraction, viz. $\frac{3x-y}{x-y}$, is 2;

$$\therefore \text{ the asymptotes are } (x+y)^2-(x+y)-2=0,$$

i.e.

$$x+y+1=0 \text{ and } x+y-2=0.$$

N.B.—If $x+y$ had not been a factor of the terms of highest degree but one in the equation, the asymptotes parallel to $x+y=0$ would have been at an infinite distance, and therefore useless. It is usual to say in this case that the curve has no linear asymptotes in that direction. The simplest case of this sort is the parabola $y^2=4ax$, where the highest degree term indicates two asymptotes in the direction $y=0$, but on investigation they are found to be at an infinite distance from the origin. We shall recur to this again under the head of curvilinear asymptotes.

The reason why the above difficulty did not arise in the case of any of the specimen curves of Art. 198 was because Nos. 4 and 5, which had repeated factors in the terms of the n^{th} degree, had no terms of the $n-1^{\text{th}}$ degree.

204. Description of the general method of finding linear asymptotes.

The general method of obtaining the equations of the linear asymptotes may thus be described.

Let the equation of the curve be arranged according to the degrees of the various terms, and let the terms of the r^{th} degree be denoted by u_r. Then the equation is of the form

$$u_n + u_{n-1} + u_{n-2} + \dots + u_1 + u_0 = 0.$$

Now (1) let $ax+by$ be a factor (unrepeated) of u_n, so that

$$u_n \equiv (ax+by)\, v_{n-1}.$$

Then, dividing by v_{n-1}, we have

$$ax+by+\frac{u_{n-1}}{v_{n-1}}+\frac{u_{n-2}}{v_{n-1}}+\dots=0.$$

The asymptote is $ax+by+c=0$, where c is the value of $\frac{u_{n-1}}{v_{n-1}}$ when $ax+by=0$ (Art. 174).

The remaining fractions all become zero when x and y are infinite.

(2) Let $(ax+by)^2$ be a factor of u_n, so that

$$u_n \equiv (ax+by)^2\, v_{n-2}.$$

Then

$$(ax+by)^2+\frac{u_{n-1}}{v_{n-2}}+\frac{u_{n-2}}{v_{n-2}}+\begin{pmatrix}\text{terms which vanish}\\ \text{when } x \text{ and } y \text{ are infinite}\end{pmatrix}=0.$$

Now $\frac{u_{n-1}}{v_{n-2}}$ is in general infinite, and there will be no linear asymptotes in the required direction unless either $ax+by$ is a factor of u_{n-1} or the terms represented by u_{n-1} are non-existent, but in that case, the method of approximation gives an equation of the form $(ax+by)^2+\mathrm{B}(ax+by)+\mathrm{C}=0$, giving two parallel asymptotes in the required direction.

[NOTE.—B will be zero if the u_{n-1} terms are non-existent, and C will be zero if the u_{n-2} terms are non-existent.]

Similarly, if u_n contains $(ax+by)^r$ as a factor, there will be no linear asymptotes (*i.e.* none at a finite distance) unless

u_{n-1} contains the factor $(ax+by)^{r-1}$, or is non-existent,
and also u_{n-2} „ „ „ $(ax+by)^{r-2}$, „ „
and so on.

If these conditions are fulfilled, there will be r parallel asymptotes given by an equation of the form

$$(ax+by)^r + \mathrm{B}(ax+by)^{r-1} + \mathrm{C}(ax+by)^{r-2} + \ldots = 0,$$

where any of the coefficients B, C,... will be zero if the corresponding terms are non-existent in the equation of the curve. Of course even in this case some, or even all (if r is even), of the asymptotes may be *imaginary*: but this is quite different from the failure by reason of any of the terms becoming infinite.

If all the asymptotes are imaginary the curve does not go to infinity at all in the specified direction; whereas if a term becomes infinite, the curve *does* go to infinity, but with no linear asymptote, just as a parabola does in fact.

This gives rise to the consideration of what are called curvilinear asymptotes, but before proceeding to the consideration of these, we will approach the theory of linear asymptotes in another way which will be instructive.

205. Intersections of a straight line and curve. Another method of finding asymptotes.

A straight line cannot cut a curve of the n^{th} degree in more than n points; for, if we eliminate y by solving simultaneously the equation of the curve and the equation $\mathrm{A}x+\mathrm{B}y+\mathrm{C}=0$, we shall obtain (in general) an equation of the n^{th} degree in x whose roots will be the abscissae of the points of intersection of the line and curve. This equation has n roots, of which any even number may be imaginary. Consequently, if a line cuts the curve in less than n real points, the deficiency will be an even number. [Similarly if we eliminate x, we shall obtain an equation in y to which similar remarks apply.]

Hence, a curve of odd degree must cut every straight line in at least one point, since it cuts it in an *odd* number, of which an *even* number may be imaginary. Consequently a curve of odd degree must extend to infinity in one direction at least.

A curve of even degree may or may not extend to infinity. A curve which does not so extend is called a closed curve. If it does go to infinity it is called an open curve.

Infinite roots. If $\mathrm{A}x+\mathrm{B}y$ is a factor of the highest degree terms in the equation of the curve, the equation giving the points of intersection of the curve with the line $\mathrm{A}x+\mathrm{B}y+\mathrm{C}=0$ will be of the $n-1^{\text{th}}$ degree.* We do not say in this case that there are only $n-1$ roots, but that there are n roots, of which one is infinite. (For the only way in which an equation of the n^{th} degree can have an infinite root is by the coefficient of its highest term being zero.) Hence in this case every line parallel to $\mathrm{A}x+\mathrm{B}y=0$ cuts the curve in one point at infinity.

If we choose C so that the coefficient of the next highest term is zero, we shall have a line which cuts the curve in two points at infinity (or possibly more). This line will be the asymptote in this direction. [See also Art. 207a on p. 249.]

By solving the equation giving the finite points of intersection of the asymptote with the curve, we shall, in general, find $n-2$ points, of which an even number may be imaginary.

If $(\mathrm{A}x+\mathrm{B}y)^2$ is a factor of the highest degree terms in the equation of the curve, and if also $\mathrm{A}x+\mathrm{B}y$ is a factor of the terms of the $n-1^{\text{th}}$ degree (or if these terms are not present), the equation giving the points of intersection of $\mathrm{A}x+\mathrm{B}y+\mathrm{C}=0$ with the curve will be of the $n-2^{\text{th}}$ degree, and every line parallel to $\mathrm{A}x+\mathrm{B}y=0$ will meet the curve in two points at infinity. If we then choose C so that a third root is infinite, which in general is possible for two values of C, we shall have two parallel asymptotes, each of which cuts the curve in at least three points at affinity.

If $(\mathrm{A}x+\mathrm{B}y)^2$ is a factor of the highest degree terms, but $\mathrm{A}x+\mathrm{B}y$ is not a factor of the terms of the $n-1^{\text{th}}$ degree we shall have no asymptotes at a finite distance. This case gives rise to curvilinear asymptotes, which we shall investigate in the next article.

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*In some cases the equation will be of lower degree, say of degree $n-r$. In this case every line parallel to $\mathrm{A}x+\mathrm{B}y=0$ meets the curve in r points at infinity, and the asymptotes will meet it in $r+1$ points, or possibly more.

EXAMPLES.

1. Apply the above method to find the asymptotes to the curves of Art. 198, and to find where the asymptotes cut their respective curves.

2. In the curve $(ax+by+c)u_{n-1}+u_{n-2}+$terms of lower degree$=0$, show that $ax+by+c=0$ is an asymptote unless u_{n-1} contains $ax+by$ as a factor.

3. Show that the asymptotes of

$$(x-y+1)(x+y+1)(x-2y+3)=\mathrm{A}x+\mathrm{B}y+\mathrm{C}$$

are $$x-y+1=0,\ x+y+1=0,\ x-2y+3=0.$$

4. Show that, in general, if a curve of the 3rd degree has three real asymptotes in different directions, the points in which the asymptotes cut the curve again are collinear.

5. Find the line in which the curve $x^3-xy^2+y^2-4=0$ cuts its asymptotes. [Find the compound equation of the asymptotes, and solve with the equation of the curve.]

206. Curvilinear asymptotes. In some (not infrequent) cases it will be found that the method of finding the asymptotes in some particular direction gives an infinite value to the constant, instead of a finite one. In such cases there are no straight line asymptotes in that direction. The simplest case is that of the parabola $y^2=ax$, which as we know goes to infinity, but has no asymptote. Another case is $(y-x)^2=a(x+y)$, which is also a parabola, but with a slanting axis.

In such cases we shall usually be able to find some simple curve (*e.g.* a parabola), which will be asymptotic to the given curve in the required direction, that is, which will approach infinitely close to the given curve when x and y are infinite. Such asymptotic curve is called a curvilinear asymptote. The drawing of the simpler, asymptotic, curve will much facilitate the drawing of the required curve.

The discovery of curvilinear asymptotes can be facilitated greatly by making use of the diagram of powers explained in Art. 178. This diagram will also serve to indicate linear asymptotes, and is of great use in many ways.

We will illustrate by an example.

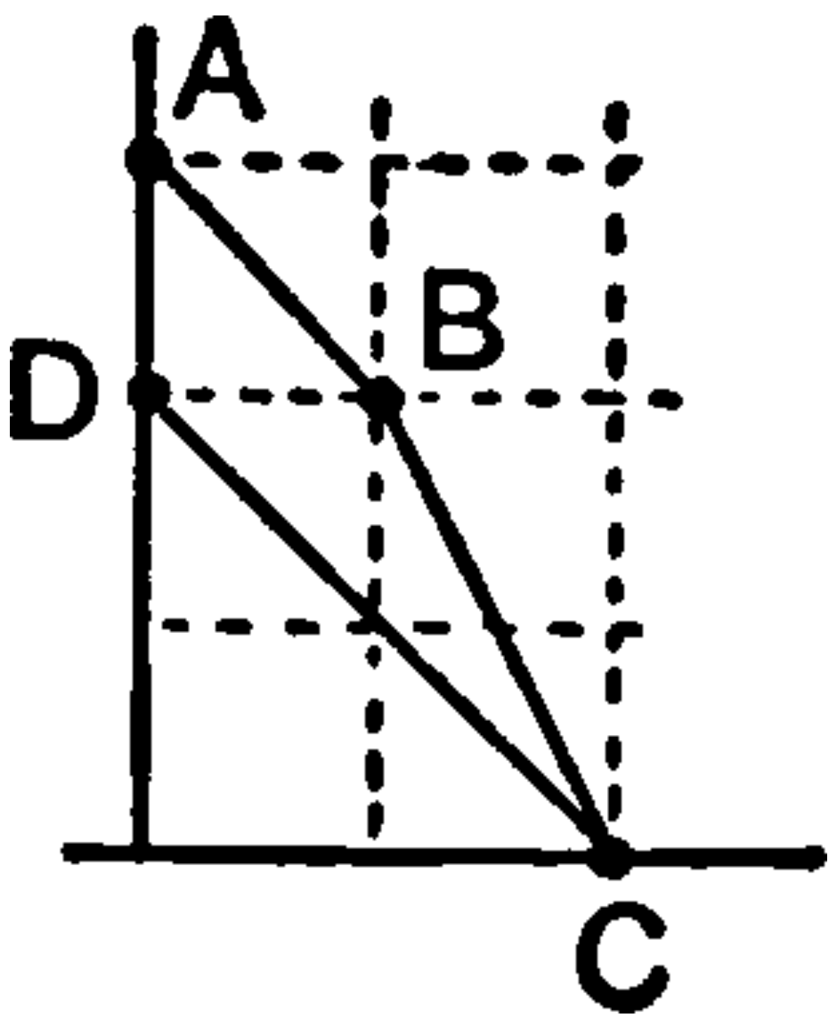
Let the equation be $y^2(2y+3x)=4y^2+9x^2$.

On consulting the diagram of powers, we see that, when x and y are infinite,

(1) y^3 may be proportional to xy^2 (see line AB), *i.e.* y proportional to x.

This leads to the linear asymptote, viz.

$$2y+3x=\left.\frac{4y^2+9x^2}{y^2}\right]_{\substack{x=2\\y=-3}}=8.$$



(2) xy^2 may be proportional to x^2 (see line BC), *i.e.* y^2 proportional to x, the first approximation being, therefore, $y^2=3x$.

Hence the equation, with the terms arranged according to degree, and the terms of each degree arranged in order of importance for this relation between y and x, is

$$y^2(3x+2y)=9x^2+4y^2,$$

i.e.
$$y^2=\frac{9x^2+4y^2}{3x+2y}.$$

By actual division, and approximating in the third stage of the division by making use of the relation $y^2:3x=1$ (which we have seen holds, for this direction, when x and y are infinite), we find $y^2=3x-2y+8$, neglecting terms which vanish when x and y are infinite.

$$\begin{array}{r}3x+2y\;)\;9x^2+4y^2\;(\;3x-2y+8\\ \underline{9x^2+6xy}\\ -6xy+4y^2\\ \underline{-6xy-4y^2}\\ 8y^2\end{array}$$

This equation, therefore, represents a curvilinear asymptote.

By arranging it in the form

$$(y+1)^2=3(x+3)$$

we see that it is a parabola with vertex at $(-3,\ -1)$, and with the horizontal axis $y+1=0$, and can easily draw it.

NOTE.—In the above example, the repeated factor of the highest degree terms which gave the direction of the curvilinear asymptote was the simple term y^2.

If the repeated factor were of the form $(ax+by)^2$, or, more generally, $(ax+by)^r$, the work would not be so simple, but by the artifice of putting $ax+by=z$, and arranging the equation in terms of x and z, or y and z, the same method would be applicable. This is true whether the factor corresponds to a curvilinear asymptote or to parallel asymptotes, though in the latter case the artifice is not needed, as the method given in previous articles is simpler and more straightforward. The student might, however, verify its applicability by putting $x+y=z$ in the equation of Art. 203, and working with x and z by substituting $y=z-x$ in the equation. He will find the division method gives $z^2=z+2$ as the equation of the asymptotes. Or he may work with y and z by substituting $x=z-y$, which will lead to the same result.

EXAMPLES.

1. Find the points in which the curve $y^2(2y+3x)=4y^2+9x^2$ cuts its linear and its curvilinear asymptote.

2. Find at what points the gradient of the same curve is (1) zero; (2) infinite; (3) equal to that of the linear asymptote.

3. Show that the origin is a conjugate point of the same curve.

4. Show that no other part of the above curve lies to the left of or below the line $3x+2y=4$.

5. Show that the line $3x+2y-4=t^2$ cuts the curve in points given by the coordinates $\left\{\frac{t(t^2+4)}{3(t+2)}, \frac{t^2+4}{t+2}\right\}$.

6. Draw the curve.

207. General hints for drawing a curve whose equation is given in rectangular coordinates.

(1) Find its linear and curvilinear asymptotes, if it has any.
(2) If it goes through the origin, find its approximate shape there.
(3) Find at what points the gradient is zero or infinite. [If the curve is a closed curve, we obtain by this means a rectangle enclosing the curve.]
(4) Note whether there are any obvious lines of symmetry.
(5) Note whether the origin is a centre of symmetry.
(6) Find any special points on the curve: *e.g.* where it cuts the coordinate axes, and where it cuts its asymptotes. Find the gradients at such points.
(7) If y can be expressed as an explicit function of x, or *vice versa*, or if x and y can both be expressed as explicit functions of a third variable, make a table of values of x, y, and gradients.

N.B.—Plot each piece of information as it is obtained, and so gradually arrive at the knowledge of the shape of the curve. Some practice is necessary, as experience makes a little information go a long way. The student will find that when an algebraic curve has linear asymptotes there are always two branches approaching each asymptote. Most frequently these two branches approach the asymptote on opposite sides and at opposite ends, as, for instance, in the case of a hyperbola; but this will not always be the case.

Sometimes a curve may be derived from a simpler curve by some simple geometrical construction: a few instances will be given in the following examples.

Drawings of the curves of Arts. 198 and 206 will be found at the end of the book. The student should compare these carefully with his own attempts at drawing the same curves, and carefully verify his work, before proceeding to the exercises below. His drawings should be on a larger scale than those in the book.

EXAMPLES.

Draw the following curves:

1. $y^2(x-a)+a^2x=0$. **2.** $y^2(x-a)-a^2x=0$.

3. $y^2(2a-x)=x^3$. [This curve is called the *cissoid*.]

4. $y^2(x-2a)=x^3$. **5.** $y^3=x^2(x-3a)$.

6. $a^2x=y(x-a)^2$. **7.** $(x+a)y^2=x^2(a-x)$.

8. $(x+a)y^2=x^2(x-a)$. **9.** $xy^2=a^2(x-a)$.

10. $xy^2=a^2(a-x)$. [This curve is called the *witch*.]

11. $ay^2=(x-a)^3$. [Note that this is the same as $ay^2=x^3$, but with every point shifted to the right through the distance a.]

12. $ay^2=x(x-a)(x-2a)$. [Show that it has a curvilinear asymptote $ay^2=(x-a)^3$, and that there is no part of the curve between $x=a$ and $x=2a$.]

13. $a^2y=x(x-a)^2$.

14. $ay(a-x)=x^3$.
Show that it has a parabolic asymptote $x^2+a(x+y)+a^2=0$.

15. $ay^2=x(x-a)^2$. **16.** $x^3-y^3=3axy$.

17. $a^3y^2=x^3(a^2-x^2)$.

18. Show that in the equation of Ex. 10, we may put

$$x=a\cos^2\theta,\quad y=a\tan\theta.$$

Hence show that, if a line be drawn through the origin to cut the circle $x^2+y^2=ax$ in P, and the line $x=a$ in Q, the point which has the abscissa of P and the ordinate of Q is on the locus. Plot the locus by this method.

19. Draw the curve $(x-a)^2y^2=x^3(2a-x)$. [This curve is called the *conchoid*.]

20. Show that, in the conchoid (Ex. 19), if $y=x\tan\theta$,

$$x-a=\pm a\cos\theta.$$

Hence show that if a line be drawn through the origin in any direction, it cuts the curve in two points P, Q, such that $PQ=2a$, and the middle point of PQ is on the line $x=a$.

Plot the curve by this method.

21. Plot the curves

$$(1)\ a^2y^2=x^2(a^2-x^2),$$
$$(2)\ a^4y^2=x^4(a^2-x^2).$$

22. If PN is the ordinate of a point P on the circle $x^2+y^2=a^2$, and if PM be drawn parallel to the axis of x to meet the line $x=a$ in M, show that the point P_1 where PN is cut by OM (O being the origin) is on the first curve of Ex. 21, and that a similar construction starting from P_1 (instead of P) will give a point on the second curve of Ex. 21. Plot the curves by this method.

23. If the point P is on the curve $y=f(x)$, and the points P_1 and P_2 are obtained by the method of last article, show that the locus of P_1 is the curve $ay=xf(x)$, and the locus of P_2 is the curve $a^2y=x^2f(x)$.

24. Show that if a line drawn from any point (x, y) to the point $(x+a, 0)$ cuts the line $x=a$ in a point M, OM will pass through the point $\left(x, \frac{x^2y}{a^2}\right)$.

25. Draw the cycloid $x=a(\theta-\sin\theta)$, $y=a(1-\cos\theta)$. Show that the tangent and normal at any point of this curve pass respectively through the points $(a\theta, 2a)$ and $(a\theta, 0)$.

26. Draw the curve $x(x-y)^2=2a^2(x+y)$.

27. Draw the curve $x(x-y)^2=a(x+y)^2$.

28. Draw the curves $x^2y^2=a^2(x^2\pm y^2)$.

Show that, if (x, y_1) and (x, y_2) are points of these curves respectively,

$$\frac{1}{{y_1}^2}+\frac{1}{{y_2}^2}=\frac{2}{a^2};$$

provided x^2 is not less than a^2.

29. Show that the eqnations $y(a^2+x^2)=ax^2$, and $y(a^2+x^2)=a^3$, both represent the curve of Ex. 10, in different positions.

30. Draw the curve $x^4-5x^2y^2+4y^4=a^4$.

31. Draw the curve $\sqrt{\frac{x}{a}}+\sqrt{\frac{y}{b}}=1$.

32. Draw the curve $\left(\frac{x}{a}\right)^{\frac{1}{3}}+\left(\frac{y}{b}\right)^{\frac{1}{3}}=1$.

33. Draw the curve $\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}=1$.

34. Draw the curve $(x-2)y^2=(y-1)x^2$.

207a. Linear asymptotes parallel to the axes. By the method of Art. 205 we see that, if there is an asymptote parallel to the axis of y, every line in the direction $x=$ constant will meet the curve in at least one point whose ordinate is infinite. Hence, if we put $x=c$ in the equation of the curve, the resulting equation in y must have fewer than n finite roots, *i.e.* the highest power of y in the equation must be of less than the n^{th} degree. If we choose such value or values for x as will make the coefficient of this highest power of y vanish, we shall obtain the asymptotes themselves, since in such case there will be still fewer finite roots to the resulting y equation. Similarly any asymptotes parallel to the axis of x will be obtained by equating to zero the coefficient of the highest power of x in the equation of the curve.

Take for example the equation

$$x^2y^2-4x^2-9y^2+3x-2y=6.$$

In this equation the coefficient of the highest power of y is x^2-9, and the coefficient of the highest power of x is y^2-4, hence the asymptotes parallel to the axes are $x\pm 3=0$ and $y\pm 2=0$.

The student can easily practise the use of this quick method of reading off such asymptotes by studying the curves given in the foregoing set of examples

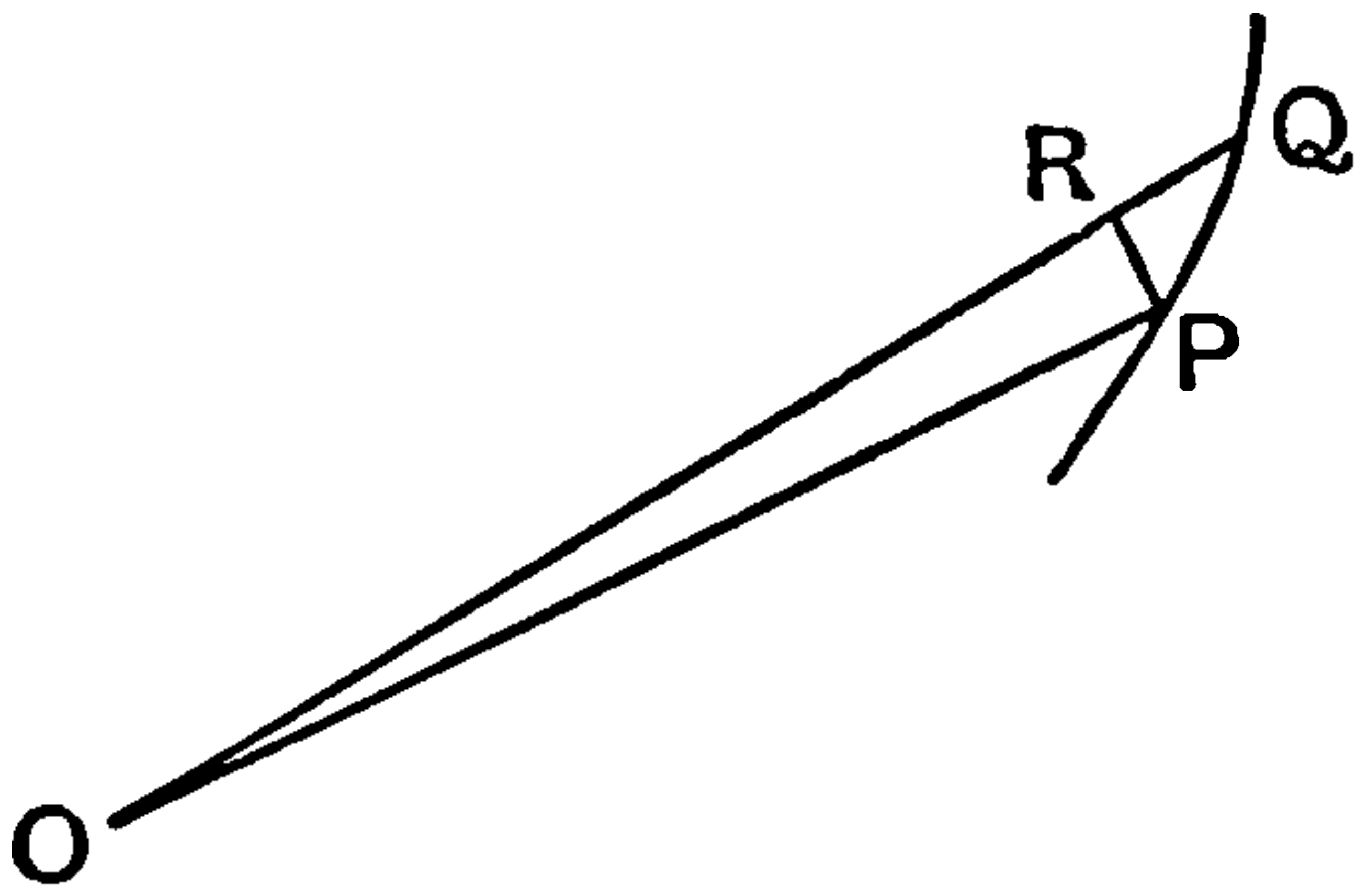
207b. Imaginary asymptotes. The student may have noticed that in many cases when on first inspection there appear to be *two* asymptotes in any given direction they turn out to be imaginary [v. Ex. (2) on p. 239]. It is usual merely to say in this case that the curve does not stretch to infinity in this direction. This as a fact is quite true, but is hardly a complete statement, for as the number of finite points in which any line in this direction cuts the curve is less than n, there must be infinitely distant points in sufficient numbers to make up the full total. Hence, the correct algebraical statement is that, in the case supposed, the infinitely distant points are imaginary.

CHAPTER XV.

POLAR COORDINATES.

208. Inclination of tangent to radius vector. In tracing a curve whose equation is given in polar coordinates, all that is necessary is to make a table of values of r and θ, and to plot the corresponding points of the curve. The labour, however, will often be much diminished, and the accuracy of plotting increased, if we can also find the direction of the tangent at each of the points so found. This is done, not by finding an expression for the gradient of the tangent, but by finding the angle made by the tangent with the radius vector to the point of contact. Let ϕ denote this angle: we shall find an expression for $\tan\phi$.

Let P, Q be two adjacent points on the curve, with the coordinates (r, θ) and $(r+dr, \theta+d\theta)$, respectively.



Draw, OP, OQ and from OQ cut off $\mathrm{OR}=\mathrm{OP}$; join PR.

Then, evidently, $\mathrm{RQ}=dr$, and, ultimately, $\mathrm{PR}=rd\theta$ (Art. 32); therefore

$$\tan(\mathrm{OQP})=\frac{rd\theta}{dr}, \text{ ultimately.}$$

Now evidently the angle OQP is, in the limit, the angle which the tangent at P makes with OP, *i.e.* $\mathrm{OQP}=\phi$, ultimately. Hence,

$$\boldsymbol{\tan\phi=\frac{rd\theta}{dr}}.$$

If we denote $\dfrac{dr}{d\theta}$ by r', the formula becomes $\tan\phi=r\div r'$.

It is often convenient to have a symbol for the reciprocal of r. It is usual to denote it by u. It is easy to see that $\tan\phi = -\frac{u}{u'}$, where u' denotes $\frac{du}{d\theta}$. For

$$u=\frac{1}{r}; \quad \therefore\ u' = -\frac{r'}{r^2}; \quad \therefore\ \frac{u'}{u} = -\frac{r'}{r}.$$

209. Polar equations of straight lines and circles. It is convenient to recognize the polar equations of straight lines and circles at a glance. Their chief forms are

(1) Straight line $\theta=0$ (called the initial line),

$r=a\sec\theta$ (perpendicular to the initial line),

$\theta=\alpha$ (any line through the origin),

$u=A\cos\theta+B\sin\theta$ (any line not through the origin).

(2) Circle $r=a$,

$r=a\cos\theta$,

$r=a\sin\theta$,

$r=a\cos\theta+b\sin\theta$,

$r^2-2r(a\cos\theta+b\sin\theta)+c=0$.

The student should draw these for different values of the constants. He should also find their Cartesian equations by writing $x=r\cos\theta$, $y=r\sin\theta$, $x^2+y^2=r^2$.

EXAMPLES.

1. Show that, in any curve, the tangent is at right angles to the radius vector when $r'=0$.

Find for what value of θ the radius vector goes through the centre of the circle $r=a\cos\theta+b\sin\theta$, and find the diameter of the circle.

2. Show that, in the equation $r^2-2r(a\cos\theta+b\sin\theta)+c=0$, r is a maximum or minimum when $\tan\theta=b\div a$, and show that the centre of the circle is at the distance $\surd(a^2+b^2)$ from the origin.

3. Show that the length of the tangent from the origin to the same circle is $\surd c$, and that the radius of the circle is $\surd(a^2+b^2-c)$.

4. In the curve $r=a(1+\cos\theta)$ show that $\tan\phi=-\cot\frac{1}{2}\theta$. Draw the curve, which is called the cardioid.

5. Show that if the equation of a curve is of the form $f(r, \cos\theta)=0$, the line $\theta=0$ is an axis of symmetry.

6. If the equation is of the form $f(r, \sin\theta)=0$, the line $\theta=\frac{\pi}{2}$ is an axis of symmetry.

7. If the equation contains only even powers of r, the origin is a centre of symmetry.

8. Draw the curve $r^2=a^2\cos^2\theta+b^2\sin^2\theta$.

9. Draw the curve $r^2=a^2\cos 2\theta$. [This is called a lemniscate.]

10. Show that, in any curve, when the ordinate is a maximum or minimum, $\tan\phi=-\tan\theta$.

Exemplify in the case of the curves $r=a(1+\cos\theta)$ and $r^2=a^2\cos 2\theta$.

11. Show that, when the abscissa is a maximum or minimum,

$$\tan\phi=\cot\theta.$$

Exemplify in the case of $r=a(1+\cos\theta)$.

12. Draw the curve $r=a\cos\theta\cos 2\theta$.

13. Draw the curve $r=a\sin^3\frac{1}{3}\theta$.

14. Show that, when the focus is the pole, the equation of a parabola can be put in the form $r(1-\cos\theta)=2a$.

Prove that $\tan\phi=-\tan\frac{1}{2}\theta$ at each point of the parabola.

15. Obtain the equation of a conic in the form

$$r=\frac{l}{1-e\cos\theta}.$$

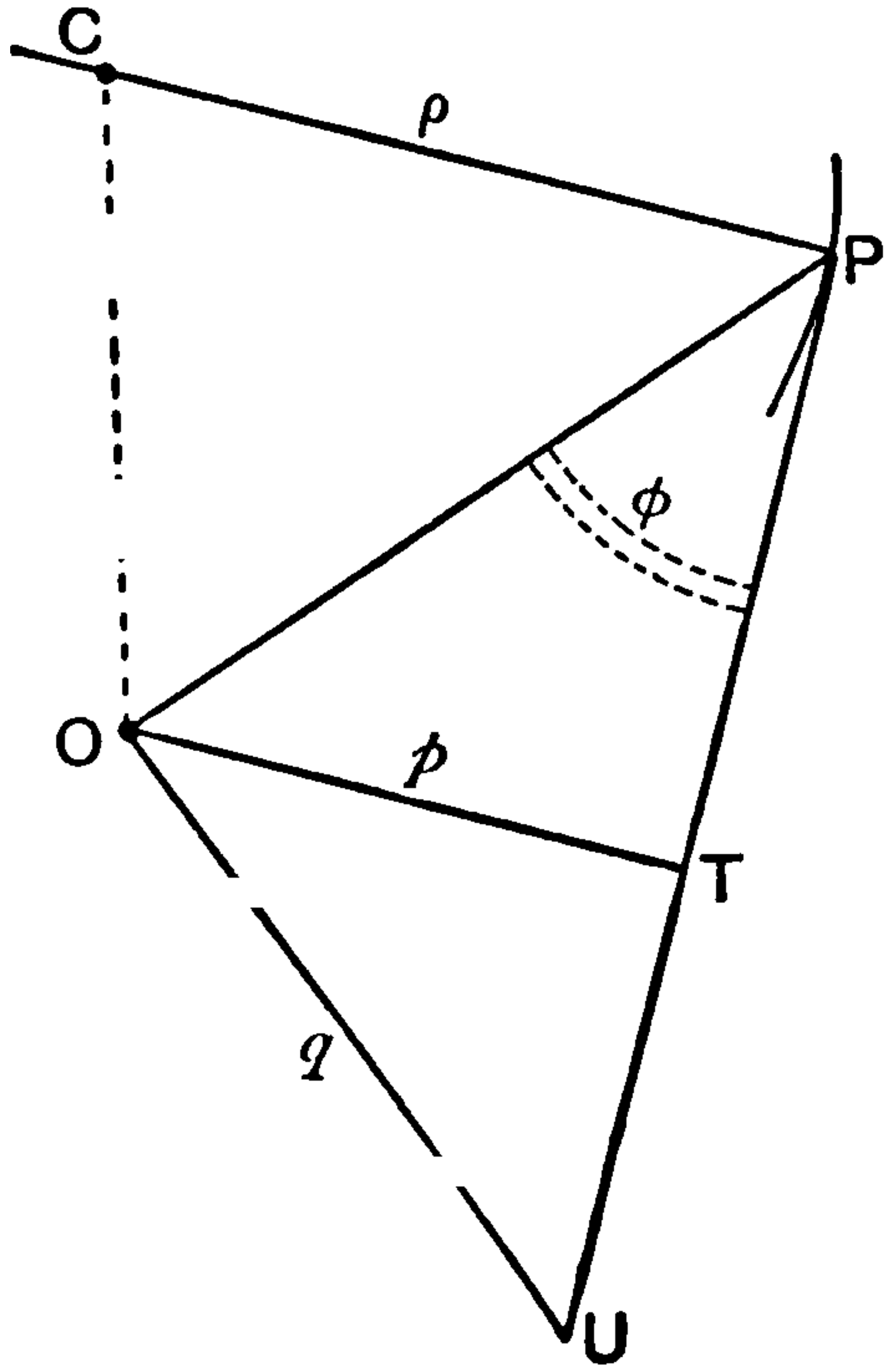
Prove that, at the highest point of the curve, $\cos\theta=e$.

16. Show that $u=\mathrm{A}+\mathrm{B}\cos\theta$ represents a parabola if $\mathrm{B}^2=\mathrm{A}^2$, an ellipse if $\mathrm{A}^2>\mathrm{B}^2$, and a hyperbola if $\mathrm{B}^2>\mathrm{A}^2$.

17. Draw the curve $r^3=a^3\sin 3\theta$.

18. Draw the curve $r=a\theta$ [the spiral of Archimedes].

210. Perpendicular drawn from the origin to the tangent.



Let p denote the length of the perpendicular OT drawn to the tangent at P.

Then, evidently,

$$\mathbf{p=r\sin\phi}.$$

The equation giving p may be put into different forms.

Thus
$$\frac{1}{p^2}=\frac{1}{r^2}\operatorname{cosec}^2\phi$$
$$=\frac{1}{r^2}\left(1+\frac{r'^2}{r^2}\right)$$
$$=\frac{1}{r^2}+\frac{r'^2}{r^4}$$

This may also be written in the form

$$\frac{1}{p^2}=u^2+u'^2$$

where u denotes $\frac{1}{r}$, and u' denotes $\frac{du}{d\theta}$, as before.

211. Pedal equation of a curve. If θ is eliminated from the polar equation of the curve by means of the above equation giving the value of p, an equation is obtained which gives the relation between p and r for every point on the curve. This equation is called the pedal equation of the curve. It is useful in connection with questions of curvature, and will be referred to again.

212. Polar sub-tangent. The line drawn from the origin to the tangent at any point, in the direction perpendicular to the radius vector of the point, is called the polar sub-tangent of the point. If the length of the polar sub-tangent is denoted by q, it evidently satisfies the equation

$$q=r\tan\phi,$$

which may be put into either of the forms,

$$q=r^2\div r', \quad \frac{1}{q}=-\frac{du}{d\theta}=-u'.$$

This last is usually the most convenient to use in connection with finding the positions of asymptotes (see below).

213. Tangents at the origin. If the curve goes through the origin, there must be some value or values of θ which make $r=0$. These values of θ will evidently give the direction or directions in which the curve goes through the origin.

214. Asymptotes. If the curve goes to infinity, there will be some value or values of θ which will make r infinite, or in other words, will make $u=0$. These values of θ give the direction or directions in which the curve goes to infinity, *i.e.* the directions of the asymptotes. The actual position of each asymptote is given by the length of the polar sub-tangent q (which is easier to find than p, and, when r is infinite, coincides with p).

It is evident from the formulae of Art. 212 that q is positive when r' is positive, *i.e.* when r increases with θ. From this it is easy to see that if we stand at the origin looking in the direction, parallel to the asymptote, for which r is positive, q must be drawn to the right if it is positive, and to the left if it is negative.

[NOTE.—It is often a good plan to transform the equation to rectangular coordinates, and to find the asymptotes by the method given in the last chapter.]

If q is infinite (*i.e.* if $u'=0$) for any value of θ which makes $u=0$, there is no *linear* asymptote in that direction.

There will be a curvilinear asymptote, but polar coordinates are not very convenient for finding curvilinear asymptotes, and it is not usually very important to do so, as the curve itself can in such cases be, as a rule, sufficiently well drawn by means of the table of values of r and θ.

215. Circular asymptotes. In some polar curves, r tends to a finite value, or to zero, when θ is infinite. In such cases the curve goes round the origin an infinite number of times, gradually approaching a circle whose centre is at the origin and whose radius is the limiting value of r. Such a circle is called an asymptotic circle or a circular asymptote. If the limiting value of r is zero, the asymptotic circle reduces to a point circle at the origin.

The curves $r\theta=a$ and $(r-c)\theta=a$ are illustrations of such cases.

EXAMPLES.

1. Find the pedal equation of the circle $r=a\cos\theta$. Prove its truth geometrically.

2. Find the pedal equation of the parabola $2au=1-\cos\theta$. Prove its truth geometrically.

3. Trace the curve $r=a\sec\theta+b$.

4. Trace the curve $r=a\sec\theta+b\tan\theta$.

5. Trace the curve $16u=5-3\cos\theta$.

6. Trace the curve $16u=3-5\cos\theta$.

7. Trace the curve $r=\frac{a\theta^2}{\theta^2+1}$, the value of θ being in radians.

8. Trace the curve $r\cos\theta=a\cos 2\theta$.

9. Trace the curve $r\cos 2\theta=a$.

10. Trace the curve $r^2\cos^2\theta=a^2\cos 2\theta$.

11. Trace the curve $r^2\cos^2\theta+a^2\cos 2\theta=0$.

12. Trace the curve $r=\frac{a\theta^2}{\theta^2-1}$.

13. Trace the curve $r=ae^{k\theta}$.
Show that the angle between the tangent and radius vector is constant. [The curve is called the equiangular, or logarithmic, spiral.] Show that different values of a give the *same* spiral, only altering its position.

216. To find the radius of curvature.

Referring to the figure, p. 251, let C be the centre of curvature corresponding to P, so that $\mathrm{CP}=\rho$.

Now, in considering the curvature, we may replace the curve by the circle of curvature for an infinitely small change in the position of P; *i.e.* dr, $d\theta$, and dp will have the same relative values for the circle of curvature as for the curve itself (for dp depends on $d\phi$, and the very essence of the circle of curvature is that $d\phi$ shall be the same for it as for the curve).

Hence, if we change (r, θ) into $(r+dr, \theta+d\theta)$ we may consider ρ and C as unchanged by considering them as belonging to the circle of curvature.

Now
$$\begin{aligned}\mathrm{OC}^2&=\mathrm{CP}^2+\mathrm{OP}^2-2\mathrm{CP}.\mathrm{OP}\cos\mathrm{OPC}\\&=\rho^2+r^2-2\rho r\sin\phi\\&=\rho^2+r^2-2\rho p.\end{aligned}$$

Now change r into $r+dr$, and let p become $p+dp$: we shall have
$$\mathrm{OC}^2=\rho^2+(r+dr)^2-2\rho(p+dp);$$
∴, by subtraction,
$$0=2r\,dr-2\rho\,dp,$$
i.e.
$$\rho=\frac{r\,dr}{dp}.$$

From this simple formula ρ can be easily calculated if the pedal equation of the curve is known.

217. Of course OC and ρ *do* change as we go from point to point *on the curve.* The above investigation shows that infinitesimal changes in r and p may be considered, regardless of the induced changes in OC and ρ, by the simple artifice of considering the circle of curvature instead of the actual curve itself. Hence, if we do consider the changes in all the quantities involved in the above equation $OC^2=\rho^2+r^2-2\rho p$, we see that there must be two *independent* differential equations, viz.

$$OC.d(OC)=(\rho-p)d\rho, \qquad \text{......................(1)}$$

and

$$rdr=\rho dp. \qquad \text{...........................(2)}$$

The first of these is easily proved geometrically if we realize that, to the first order of small quantities, the new centre of curvature must lie on CP; for then $d(OC)\div d\rho=\cos OCP$ (as can be seen at once from a figure), and so also is $(\rho-p)\div OC$. The second of them is the equation giving the value of ρ.

EXAMPLES.

1. Draw the parabola $r=\dfrac{2a}{1-\sin\theta}$.

Find its pedal equation, and deduce the length of the radius of curvature at any point.

Find the value of ρ at (1) the vertex; (2) the ends of the latus rectum.

2. Find the radius of curvature at any point of the curve $r^2=ap$.

3. Find the pedal equation of the cardioid $r=a(1+\cos\theta)$, and deduce the radius of curvature at any point.

4. Find the pedal equation of the curve $r^2=a^2\cos 2\theta$, and the radius of curvature at any point.

5. Find the radius of curvature at any point of the curve $r^n=a^n\cos n\theta$. Show that the cardioid and the parabola are particular cases of this curve.

6. If ϕ is the angle between the radius vector and the tangent at the point (a, α) of a curve, show that the equation of the tangent is

$$\frac{a}{r}=\frac{\sin(\phi-\theta+\alpha)}{\sin\phi}=\cos(\theta-\alpha)-\cot\phi\,.\sin(\theta-\alpha),$$

and that of the normal is

$$\frac{a}{r}=\frac{\cos(\phi-\theta+\alpha)}{\cos\phi}=\cos(\theta-\alpha)+\tan\phi\,.\sin(\theta-\alpha).$$

MISCELLANEOUS EXAMPLES.

1. If y is a function of x, explain the meaning of $\dfrac{dy}{dx}$. Find, from your definition, the values of $\dfrac{dy}{dx}$ when

$$(1)\ y=3x^4;\quad (2)\ y=\sin 2x.$$

2. Differentiate (1) $\dfrac{2x^2+3x+1}{2x^2-3x+1}$;

$$(2)\ x\sqrt{x^2+1}+\log_e(x+\sqrt{x^2+1}).$$

3. From the definition of a differential coefficient, deduce that $\dfrac{d(a^x)}{dx}$ is proportional to a^x, if a is a constant.

Give the value of $\dfrac{d(10^x)}{dx}$ when $x=3{\cdot}46$, and compare it with the value of $\dfrac{10^{3\cdot47}-10^{3\cdot46}}{0{\cdot}01}$, accounting for the similarity of the results. [C.S.C.]

4. If the curve $y=a^x$ crosses the axis of y at an angle of 30°, find the value of a.

5. In the curve $y=e^{-ax}\sin bx$, P, Q, R are points on the axis of x, P and R being any two successive crossing points of the curve, and Q the foot of the intervening maximum or minimum ordinate.

Find the ratio of PQ to PR. When $a=1$, find the value of b for which this ratio has the value $2:5$. [C.S.C.]

6. Two particles P, Q revolving round a point O in circles of radii a, b, with linear velocities u, v respectively, start from points on the same radius, revolving in the same direction.

Find an expression for the angular velocity of PQ at any time, and the angle between OP and OQ when the direction of the angular velocity of PQ begins to reverse, showing under what velocity conditions such reversion can occur. [C.S.C.]

7. Find the greatest cylinder that can be cut from a conical block, the height of the cone being 12 feet, and the radius of its base being 5 feet.

8. A man has to go along a straight road and turn off to a point P 600 feet from the road in a field lying alongside the road. He can go twice as fast along the road as over the field. At what point must he leave the road to reach P as quickly as possible? [C.S.C.]

9. A small screen is placed perpendicularly between two lights A and B whose lighting powers are as $3:4$. The intensity of illumination of the screen by either light varies inversely as the square of the

distance of the screen from the light. A and B being 7 feet apart, find the distance from A at which the sum of its illuminations by the two lights is a minimum, and find also the points at which it will be equally illuminated by the two lights. [C.S.C.]

Draw a graph whose ordinates are proportional to the total illumination at different distances, x, measured from A.

10. Draw the curve $y=\frac{1}{10}(6x^2-x^3)$, and find the point where its radius of curvature is infinite. Show that all chords drawn through this point are bisected there.

11. The coordinates of a point on a curve are given by the equations

$$x=at+b,$$

$$y=a't-ct^2,$$

where t is a variable, and a, b, c are constants.

Show that $\frac{dy}{dx}$ is a linear function of x.

12. Show that the tangents to the curve $2x^2+3xy+4y^2=100$, at the points where it is intersected by the lines $4x+3y=0$, and $3x+8y=0$ form a rectangle whose sides are parallel to the axes of coordinates.

13. Find the directions of the lines drawn from the origin to curve $2x^2+3xy+4y^2=100$ which are respectively of maximum minimum length.

14. Find the differential coefficient of $\sin x$ with regard to x, when x is expressed in minutes.

sight, to the nearest minute, from its sine.

Show that a four figure table will suffice up to about 70°.

How far will a five figure table suffice?

[Take 3437 minutes = 1 radian.]

15. The expression $3(a+x)^4-7(a+x)^3+12$ is to be arranged in powers of x, *i.e.* in the form

$$A+Bx+Cx^2+Dx^3+Ex^4.$$

Differentiate this identity with respect to x, and in the result put $x=0$, and thus find B in terms of a.

Proceed in a similar manner to find C, D, E in terms of a. What is A in terms of a?

16. Find the equations of the tangent and normal to the curve

$$\sqrt{y}-\sqrt{k}=m(\sqrt{x}-\sqrt{h})$$

at the point (h, k).

17. The length s of a wire stretched somewhat tightly between two points in the same horizontal line, and sagging x feet at the centre, is given by the equation

$$s=\surd(a^2+4x^2)+\frac{a^2}{2x}\log_e\frac{2x+\surd(a^2+4x^2)}{a},$$

$2a$ being the distance between the points of support.

Assuming this, find $\frac{ds}{dx}$.

If $a=100$ feet, and $x=5$ feet, show that paying out an inch more wire increases the sag at the centre by $7\frac{1}{2}$ inches nearly.

Examine the limit to which the expression for s tends, when x is indefinitely small.

18. Find the condition which must be satisfied at a point of inflexion on the curve $y=f(x)$.

Show that the curves

$$y=x^3+x^2-x-1,$$

and

$$y=2(x^3-x^2+x-1)$$

touch and cross each other at the point for which $x=1$.

Find the length of their common radius of curvature at this point.

Trace the curves.

19. Find the radius of curvature of the parabola

$$(x-y)^2-2a(x+y)+a^2=0,$$

at the point where it touches the axis of x.

Show that the tangent to this curve cuts off intercepts on the axes whose sum is constant.

20. If $f(x)$ can be expanded in ascending powers of x, find the coefficient of x^n.

Expand $\frac{\cos x}{1+\sin x}$ as far as x^5.

21. Find the radii of curvature, at the origin, of the curves

$$(1)\ x+y=ax^2+by^2,$$
$$(2)\ x^2-y^2=x^3.$$

22. Show that the roots of the equation $8x^3-6x+1=0$ are the abscissae of the points of intersection of the parabola $y=4x^2$ and the hyperbola $y=3-\frac{1}{2x}$, and hence, by drawing the curves, show that the equation has two positive roots and one negative root, all less than unity.

Calculate the roots to 3 places of decimals. [C.S.C.]

23. A tree-trunk is assumed to be in the form of a solid of revolution whose axis is vertical, and the area of any horizontal section is k times the weight of the portion above that section. Prove that, if W is the weight of a unit volume of the wood, the area of the section at a height x above the ground is proportional to e^{-kWx}.

Find, in lbs. per square foot, the pressure across any horizontal section of the trunk, having given that a cubic foot of the wood weighs 56 lbs., and that the areas of the sections at the base and at a height of 10 feet are in the ratio 4 : 1. [Take $\log_e 2 = \cdot 7$.] [C.S.C.]

24. Find approximately the situations of the real roots of the equation $(x^2+1)^2 = 16000(x-10)$, and calculate them to 3 significant figures.

25. Determine the two points on the curve $4y^3 - 12y^2 + 9y + 4x = 1$ at which the tangent is perpendicular to the axis of x, and the one point of inflexion. Draw the curve.

26. Find the least amount of canvas for a conical tent whose cubic contents shall be 120 cubic feet.

27. If $y = \sin(\log x)$, prove that $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = 0$.

28. State Leibnitz's theorem and apply it to find the result of differentiating the equation $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = 0$ n times.

29. Apply Maclaurin's theorem to obtain the expansion of $\log_e(1+x)$ in positive powers of x.

For what values of x does the expansion hold?

Prove that $\log\left(\dfrac{\tan x}{x}\right) = \dfrac{x^2}{3} + \dfrac{7x^4}{90} + \dots$.

30. Given the volume of a right circular cone, prove that if $mh + nr$ is to be as small as possible, where h is the altitude of the cone, and r is the radius of its base, then $2mh = nr$.

31. Draw a diagram to represent the function $z = \sin x \sin y$, between the limits 0 to π for both x and y, by means of its contour lines for the values $z = 0$, $\cdot 25$, $\cdot 5$, $\cdot 75$ and 1.

What is the geometrical meaning of $\sqrt{\left\{\left(\dfrac{dz}{dx}\right)^2 + \left(\dfrac{dz}{dy}\right)^2\right\}}$?

Show on the diagram where this quantity has its greatest values.

32. Find the least root, other than zero, of the equation $\tan \frac{1}{2}\pi x = \sqrt{x}$.

33. A function of the 3rd degree vanishes for $x = 1$ and $x = -2$, and its values when $x = 2$ and $x = -1$ are 8 and -1 respectively. Find the function, determine its maximum and minimum values, and draw the graph.

34. If u, v are such functions of rectangular coordinates x, y that the curves $u=$constant, $v=$constant intersect at right angles, prove that if x, y are expressed as functions of u, v,

$$\frac{dx}{du}=\frac{1}{R^2}\frac{du}{dx};\quad \frac{dx}{dv}=\frac{1}{R^2}\frac{dv}{dx},$$

$$\text{where } R^2=\left(\frac{du}{dx}\right)^2+\left(\frac{du}{dy}\right)^2.$$

35. The curve whose equation is $y=xe^{-mx}$ has been suggested as giving the powder pressures in the bore of a gun.

Trace this curve, taking $m=2$, and find when y is a maximum, and where the curve has a point of inflexion.

36. Find at what points and gradients the curve whose equation is $y=x\tan\frac{\pi y}{2a}$ cuts the axes of coordinates.

Draw the curve between the limits $y=\pm 2a$.

37. Assuming that (1) the cost of a water main varies as the square of d, the internal diameter; (2) that a given quantity of water is to be steadily pumped through the main; (3) that the cost of a unit of work is constant; (4) that the work expended in pumping varies as $d\,.\,v^{\frac{3}{2}}$, where v is the velocity of the water; prove that, as regards annual expenditure, the most economical diameter of the main is that for which the interest of the cost of the main$=$the cost of the work expended in pumping.

38. The equation of a curve is

$$y=a\log_e\frac{a+\sqrt{(a^2-x^2)}}{x}-\sqrt{(a^2-x^2)}.$$

Show that the length of the tangent intercepted between the curve and the axis of y is constant.

39. A solid sphere is just totally immersed in water. Assume that the pressure of water on any horizontal plane area varies jointly as the area and as the depth of the area. Find what portion of the sphere must be cut away by a horizontal plane in order that the pressure on the plane surface of the remainder may be a maximum.

40. If the tangent at any point of a plane curve is written in the form $x\cos\alpha+y\sin\alpha=f(\alpha)$, show that the equation of the corresponding normal is $y\cos\alpha-x\sin\alpha=f'(\alpha)$.

41. If $x=3y-4y^3$, show that

$$9\left\{(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}\right\}+y=0.$$

42. If n is a positive integer, show that

$$\left(\frac{d}{dx}\right)^{n+1} x^n \log x = \underline{|n} \div x.$$

43. If there is an equation $f(p, t, v)=0$ between three variables p, t, v, show that

$$\left(\frac{dp}{dt}\right)_{v \text{ constant}} \times \left(\frac{dt}{dv}\right)_{p \text{ constant}} \times \left(\frac{dv}{dp}\right)_{t \text{ constant}} = -1.$$

44. If u, v are functions of z, and z is a function of x and y, show that

$$(1)\ \frac{du}{dx} \cdot \frac{dv}{dy} = \frac{du}{dy} \cdot \frac{dv}{dx},$$

$$(2)\ \frac{d}{dy}\left(u \frac{dv}{dx}\right) = \frac{d}{dx}\left(u \frac{dv}{dy}\right).$$

45. If $\dfrac{d^2}{dx\,dy}\left(\dfrac{z}{xy}\right)=0$, prove that

$$xy \frac{d^2z}{dx\,dy} + z = x\frac{dz}{dx} + y\frac{dz}{dy}.$$

46. Show that at points on a plane curve where the curvature is a maximum or minimum, excluding points of zero curvature,

$$3\frac{dy}{dx} \cdot \left(\frac{d^2y}{dx^2}\right)^2 = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\} \frac{d^3y}{dx^3}.$$

Show that the points of maximum curvature in the curve $y = x - x^3$ have abscissae $x = \pm\,{\cdot}6$ approximately.

47. In the curve $x = a\cos^3\theta$, $y = a\sin^3\theta$, show that the length of the tangent intercepted between the coordinate axes is constant.

48. If $x=a$ is an approximate value of the abscissa of a point of intersection of the curves $y=\mathrm{F}(x)$ and $y=f(x)$, obtain a closer approximation.

49. Evaluate $\dfrac{x - x^x}{1 - x + \log x}$ when $x=1$.

50. If u is a function of $y-z$, $z-x$, and $x-y$, show that

$$(1)\ \frac{du}{dx} + \frac{du}{dy} + \frac{du}{dz} = 0,$$

$$(2)\ 3\left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\right) = 2\left(\frac{d}{dx} + \omega\frac{d}{dy} + \omega^2\frac{d}{dz}\right)\left(\frac{d}{dx} + \omega^2\frac{d}{dy} + \omega\frac{d}{dz}\right)u,$$

where $\omega^2 + \omega + 1 = 0$, *i e.* ω is a cube root of unity.

51. If $u+iv=f(x+iy)$, where i denotes $\surd(-1)$, show that

$$(1)\ \frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}=0,$$

$$(2)\ \frac{du}{dx}\cdot\frac{dv}{dx}+\frac{du}{dy}\cdot\frac{dv}{dy}=0.$$

52. If $x^xy^yz^z$ is constant, show that, when $x=y=z$,

$$x(1+\log x)\frac{d^2z}{dx\,dy}=-1.$$

PAPERS FOR PRACTICE.

I.

1. Differentiate from first principles x^n, where n is any given constant, and write down the differential coefficients with respect to x of

$$\frac{a^2-x^2}{a^2+x^2},\ e^{-4x}\cos(3x-a),\ \log\cot\left(\frac{x}{2}+\frac{\pi}{4}\right).$$

2. A point P moves along a straight line AB, the distance AP being x feet at the time t seconds after a given instant. If

$$x=10+6t+8t^2,$$

find (1) the velocity and acceleration of P when $t=3$; (2) the space described and the velocity added on during the next second.

Compare the results in (1) and (2), explaining why one pair agree, and the other pair do not agree.

3. Prove that if the differential coefficient of a function of a variable changes its sign as the variable changes its magnitude, the function is either a maximum or a minimum, and show how to determine which it is. Find the maxima and minima values of

$$2x^5-5x^4-20x^3+13,$$

and show that $(\tan^3x\cot 3x)$ has a maximum value when

$$x=\frac{\pi}{8}.$$

4. If PM be the ordinate at any point P on the curve

$$y=\frac{c}{2}\cdot\left(e^{\frac{x}{c}}+e^{-\frac{x}{c}}\right),$$

find the equation of the normal at P, and show that the projection of MP on the normal at P is of constant length.

If the normal at P cuts the axis of x at G, so as to make $PG=4c$, find the position of P.

5. Find the maximum and minimum values of

$$\frac{(x-1)x(x+1)}{x^4-x^2+1}.$$

Also if $$y^4-xy^2+x^4=0,$$
find maxima and minima values of x, and of y.

6. If u is a function of x and y, show that

$$du=\frac{\partial u}{\partial x}.dx+\frac{\partial u}{\partial y}.dy.$$

Explain the meaning of the symbols by means of a geometrical illustration.

If x and y are connected by the relation

$$\frac{dy}{dx}=\frac{y-1}{x\log x},$$

prove that $$\frac{d^ny}{dx^n}=\frac{(-1)^{n-1}\,|\underline{n-1}}{x^n.\log x}.(y-1).$$

7. Establish the formula $$\rho=r\frac{dr}{dp},$$

where ρ is the radius of curvature, and p the perpendicular from the pole to the tangent at (r, θ) on the curve.

In the curve $r=ae^{\theta\cot\alpha}$, show that the tangent makes with the radius vector at the point of contact a constant angle, and that the centre of curvature is at the point where the normal is intersected by a line through the pole perpendicular to the radius vector.

II.

1. Define a differential coefficient, and from the definition obtain $\frac{dy}{dx}$ in the following cases:

$$\text{(i) } y=\sqrt{x}; \quad \text{(ii) } y=\sin x.$$

Indicate by a figure the meaning of $\frac{dy}{dx}$ in case (ii).

If v cubic feet be the volume of water in a tank, t seconds after a discharge pipe has been opened, give the meaning of $\frac{dv}{dt}$, and of $\frac{d^2v}{dt^2}$.

2. Differentiate with respect to x the following expressions:

$$\frac{\sqrt{x^2-5}}{x+3}, \quad (\sec x\tan 2x), \quad \sin^{-1}\frac{2x}{5}+2\tan^{-1}3x,$$

$$e^{ax}\cos(5x+2), \quad x^{\frac{3}{2}}\log\frac{x+a}{b-x}.$$

3. Show how the differential calculus is applied to obtain the maximum and minimum values of a given function. Illustrate in a diagram the fact that some of the maximum values may be less than some of the minimum values.

If
$$y=4x^5(x-3)+3(3x^4+4),$$
find what values of x make y a maximum or minimum, and give the corresponding values of y.

4. Find the functions of x of which the following expressions are the differential coefficients:

$$\text{(i)}\ (a+x)(c^2-x^2)\,;\quad \text{(ii)}\ \frac{x^2-3x+3}{x^2+2}\,;$$
$$\text{(iii)}\ \frac{2x^2+3}{x^4-5x^2+4}\,;\quad \text{(iv)}\ \frac{x}{x^4+5x^2+4}.$$

5. Show that maxima and minima values of $f(x)$ occur alternately: investigate these values when

$$f(x)=7+6x-x^3-3\tan^{-1}\left(\frac{2x}{1-x^2}\right).$$

Find the minimum triangle cut off by a line which passes through a fixed point A and intersects two fixed straight lines OB, OC: show that there is only one minimum and no true maximum.

6. Prove the formula
$$\rho=\frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}},$$
and show how the formula is modified when x and y are given in terms of an independent variable θ. In the case of the cycloid, where
$$x=a(\theta+\sin\theta),$$
$$y=a(1-\cos\theta).$$
find the radius of curvature at the vertex, and at the point where
$$\theta=\frac{\pi}{3}.$$

7. If
$$x=r\cos\theta \text{ and } y=r\sin\theta,$$
and x and y are functions of the independent variable t, show that
$$x\frac{dy}{dt}-y\frac{dx}{dt}=r^2\frac{d\theta}{dt}\,;$$
also express
$$\frac{d^2x}{dt^2},\ \text{and}\ \frac{d^2y}{dt^2}$$

in terms of r, θ, and their differential coefficients with respect to t, and find the value of

$$\frac{d^2y}{dt^2}\cos\theta-\frac{d^2x}{dt^2}\sin\theta.$$

If x, y be the co-ordinates of a moving point at time t, what is the interpretation of the results obtained above?

III.

1. Illustrate by diagrams the difference between a continuous and a discontinuous function.

What is meant by the limiting value of $f(x)$ when $x=0$? Find the limits, when $x=0$, of

$$\frac{3x}{4-x},\ \frac{x}{2x+1},\ \text{and}\ \frac{\sqrt{1+x}-\sqrt{1-x}}{x}.$$

2. Find $\frac{dy}{dx}$ in the following cases:

$$(1)\ y=\frac{1}{2x^3}.$$

$$(2)\ y=\frac{\sin x-\cos x}{\sin x+\cos x}.$$

$$(3)\ y=e^{3x}.\cos\left(4x-\frac{\pi}{3}\right).$$

3. Find the minimum values of

$$x^2-3+\frac{1}{x^2+3},$$

and the maximum value of

$$\frac{\sin x}{1+\tan x}.$$

4. State Maclaurin's theorem, and apply it to expand $\tan x$ as far as terms involving x^7.

Thence deduce

$$\tan^2x=x^2+\frac{2}{3}x^4+\frac{17}{45}x^6+\dots.$$

5. If
$$\sqrt{y}=\log[x+\sqrt{x^2+a^2}],$$

prove that
$$(a^2+x^2)\frac{d^2y}{dx^2}+x\frac{dy}{dx}-2=0,$$

and find the first five terms in the expansion of y in ascending powers of x.

6. If

$$\frac{x}{a}+\frac{y}{b}-1=0,$$

find the maximum value of xy, and the minimum value of x^2+y^2. What is the geometrical representation of these results?

Show that the area of the largest right-angled triangle of given perimeter $2l$ is $l^2(3-2\sqrt{2})$.

7. Define the curvature at any point of a curve. If S be the focus of a parabola, P a point on the curve, prove that the radius of curvature at P is equal to

$$2\sqrt{\frac{SP^3}{a}},$$

$4a$ being the latus-rectum. At what points is the curvature half that at the vertex? Give the positions of the corresponding centres of curvature.

8. Trace the curve $ay^2-x^3+3ax^2=0$ and determine the singular points on it.

Show that the tangents at the points of inflexion are given by the equations $3x-8a\pm y=0$.

IV.

1. Define the differential coefficient of a function of a variable, and show that it measures the rate of change of the function.

Given that $\sin x$ increases as x varies from 0 to $\frac{\pi}{2}$, show from the principles of the differential calculus that the rate of increase diminishes. Verify this result geometrically.

2. Differentiate with respect to x the following expressions:

$$\cos^3\left(\frac{x}{a}\right),\ \frac{1-2x}{3+x^2},\ \tan^{-1}(2x-1),$$

$$a^{2\sin x}\log\frac{\sqrt{a^2-x^2}}{a+2x}.$$

3. Two curves $yx^2=\text{constant}$, and $yx=\text{constant}$,

pass through the point P (3, 4): find the angle at which they intersect each other at P.

Find also the length intercepted on the axis of x between the two normals drawn to the curves at P.

4. Establish a method of obtaining the limiting value of a fraction

$$\frac{f(x)}{F(x)}$$

for a value of x, which makes both the numerator and the denominator to vanish. Find the limiting values of y in the following cases:

$$(1)\ y=\frac{x^x-1}{x^m\log x+x^n-1}\ \text{when}\ x=1;$$

$$(2)\ y=\sin\frac{a}{x}.\log(2+3e^x)\ \text{when}\ x=\infty;$$

$$(3)\ y=\left(\cos\sqrt{\frac{4}{x}}\right)^x\ \text{when}\ x=\infty.$$

5. If

$$y=f(x),$$

explain how $\frac{dy}{dx}$ may be used to determine whether $f(x)$ increases or decreases with x, and apply the method to the case where

$$f(x)=\sin^{-1}x,$$

and $\sin^{-1}x$ denotes an angle between 0 and $\frac{\pi}{2}$.

Show that if $\sin^{-1}x$ lies between 0 and $\frac{\pi}{2}$,

$$(1+x)\log(1+x)-(1-x)\log(1-x)-2\sqrt{1-x^2}.\sin^{-1}x$$

is positive for all values of x between 0 and 1.

6. Explain "contact of the first order" and "contact of the second order."

Find the curvature of the ellipse whose semi-axes are 3 inches and 5 inches long, at an extremity of a latus-rectum: and find the position of the corresponding centre of curvature.

Show that the circle of curvature at an extremity of either axis has with the curve contact of the third order.

7. If

$$z=f(x, y),$$

where x and y are independent variables, explain the terms: (1) partial differential coefficients of z with respect to x and y, (2) total differential of z; and give a geometrical illustration of their meaning.

If

$$z=\log(\tan x+\tan y),$$

show that

$$\sin 2x.\frac{\partial z}{\partial x}+\sin 2y\frac{\partial z}{\partial y}=2.$$

8. Trace the curve

$$x^2y-x^2+5y=0,$$

and find its points of inflexion.

V.

1. Obtain from first principles the differential coefficient of $\cos x$ with respect to x, and illustrate the method by a diagram.

The diameter of a sphere is increased from 6 inches to 10 inches in 49 hours, equal masses of homogeneous matter being added in equal times: what is the radius after $18\frac{1}{2}$ hours, and at what rate in inches per hour is the diameter then increasing?

If the mass of a sphere of 2 inches diameter is m lbs., and the moment of inertia of a solid homogeneous sphere, of mass M lbs. and radius r inches, about a diameter is $\mathrm{M}.\frac{2r_2}{5}$; at what rate is the moment of inertia increasing at this same instant?

2. Establish the formula for the differentiation of a quotient.

Find
$$\frac{d}{dx}.\frac{2-x}{\sqrt{(a^2+x^2)}},$$
$$\frac{d}{dx}.\log\left(\frac{1+x\sin x}{1-x\sin x}\right),$$
$$\frac{d}{dx}.e^{ax}\cos(bx+c).$$

Find
$$\frac{d^4y}{dx^4} \text{ when } y=\cos 2x, \text{ and}$$
$$\frac{d^3z}{dx^3} \text{ when } z=\frac{1}{(1-2x)^2}.$$

3. Give the equations of the tangent and of the normal at a point on the curve $y=f(x)$.

If the curves $y^2=4ax-4a^2$, $27ay^2=4(x-3a)^3$ be intersected at points P, Q respectively by the line $y=2a$, show that the normal at P to the first curve intersects the tangent at Q to the second curve at a point on the axis of x.

4. Write down the expansion of $f(x+h)$ in powers of h by Taylor's Theorem, and state, without proof, some form of the remainder after n terms. Expand $\sin(x+h)$ in ascending powers of h, and give the remainder after three terms. Hence show that, by taking the first three terms only, $\sin 60° \ 57{\cdot}3'$ can be found with an error less than 4×10^{-7}.

5. If $f(x)$ is a continuous function of x between $x=a$ and $x=b$, and $f(a)=f(b)$, show that there must be at least one maximum or minimum value of $f(x)$ between $x=a$ and $x=b$.

In the curve
$$y=\frac{(x-2)(x-3)}{x-4},$$
find the maximum and minimum values of y. Indicate these values and the general form of the curve in a diagram.

6. Explain what is meant by a partial differential coefficient.

The area of a triangle is calculated from the measurements of two sides and their included angle; find approximately the error in the area due to given small errors in the measurements. Find also the approximate error in the length of the third side if calculated from the given measurements.

7. Establish the formulae

$$(1)\ \tan\phi = r\frac{d\theta}{dr},$$

$$(2)\ \frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2 \text{ where } u = \frac{1}{r},$$

and show that the curves

$$r^n = a^n \sin n\theta \text{ and } r^n = b^n \cos n\theta$$

cut at right angles. Sketch the curves when $n=2$.

VI.

1. Find $\frac{dy}{dx}$ in the following cases:

$$y = \tan 4x,\quad y = \frac{2+x^2}{1-x},\quad y = xe^{-3x},\quad y = \sin^{-1}\left(\frac{x}{2a}\right).$$

2. Find the equation of the tangent to the curve $y = f(x)$ at the point $(x'y')$.

In the catenary $$y = \frac{c}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right),$$

show that the length of the perpendicular let fall from N, the foot of ordinate PN, upon the tangent at the point P, is of constant length.

Also, if the normal at P meet the axis of x at G, show that PG varies as PN^2.

3. When is a function $f(x)$ said to have a maximum value? Find the condition that $f(x)$ should have a maximum value when $x=a$, when the first of the successive differential coefficients of $f(x)$ which does not vanish when $x=a$ is the nth.

A portion of a paraboloid of revolution cut off by a plane perpendicular to the axis has a coaxial right cylinder inscribed in it. Show that the greatest possible volume of this cylinder is ½ the volume of the paraboloid.

4. Show that the curvature of a circle is measured by the reciprocal of the radius. Establish *one* of the two formulae

$$\rho=\frac{\left\{1+\overline{\frac{dy}{dx}}\Big|^{2}\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}},$$

$$\rho=r\frac{dr}{dp}.$$

Find ρ at the points where the axis of x meets the curve

$$3y^2(x+c)=x^2(3c-x).$$

5. In the curve $r=f(\theta)$ find an expression for the angle contained by the radius vector and the tangent at any point P on the curve.

Trace the curve $r=a\sec\theta\pm a\tan\theta$ and draw an asymptote.

If a radius vector OPP′ be drawn cutting the curve in P and P′, and if the tangents at P and P′ meet at T, prove that PT=P′T.

6. Separate the fraction $\frac{x^2+1}{x^3-1}$ into partial fractions, and find its integral.

7. The co-ordinates of any point on a cycloid being given in the form

$$x=a(\theta-\sin\theta),$$
$$y=a(1-\cos\theta),$$

show that the radius of curvature at any point is twice the length of the normal between the curve and the axis of x.

VII.

1. Explain what is meant by the statement that the limit of $\frac{\sin\theta}{\theta}$, when θ is indefinitely diminished, is unity; and hence deduce that the area of a circle of radius r is πr^2, and the circumference $2\pi r$.

If $y=3x^2+2$, obtain from first principles the differential coefficient of y with respect to x, and show with a diagram the geometrical meaning of your result. For what value of y is this differential coefficient equal to unity?

2. Obtain the differential coefficients with respect to x of

$$\tan^2 3x,\ x^2\log 4x,\ \text{and}\ \sqrt{\frac{a-2x}{a+3x}}.$$

If $$y=ae^{3x}\sin\left(\frac{2x}{3}+\frac{\pi}{4}\right),$$

find $$\frac{dy}{dx}\ \text{and}\ \frac{d^2y}{dx^2},$$

and show that $$\frac{d^2y}{dx^2}-6\frac{dy}{dx}+\frac{85}{9}y=0.$$

3. If $x+y=20$, find when xy is a maximum.

Also, if
$$y=2x^3-27x^2-132x+2,$$
find its maxima and minima values.

4. Assuming that $f(x)$ admits of expansion in a series of positive integral powers of x, obtain the series for $f(x)$.

Find also the first four terms in the expansion of $\sec x$ in ascending powers of x.

5. Find the limiting value of $\dfrac{f(x)}{\phi(x)}$ when $x=a$,

$$(1)\ \text{if } f(a)=0 \text{ and } \phi(a)=0;$$

$$(2)\ \text{if } f(a)=\infty \text{ and } \phi(a)=\infty.$$

Give the values of
$$\frac{x^x-x}{1-x+\log x} \text{ when } x=1,$$
and of
$$\frac{\log\tan 7x}{\log\tan 4x} \text{ when } x=0.$$

6. Define maxima and minima values of a given function of x, and show how to determine them.

The sum of the perimeters of two equal squares and a circle is 100 feet. When is the sum of the areas least, and when is it greatest? Are these really minimum and maximum values?

7. If $z=f(x, y)$ be a homogeneous function of x and y, of n dimensions, show that
$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=nz.$$

Show also that if
$$u=f\left(\frac{y}{x}\right)+a\sqrt{x^2+y^2},$$
then
$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=a\sqrt{x^2+y^2}.$$

Give the geometrical meaning of
$$\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y},$$
when $x=2$, and $y=1$, if $z^2=12-x^2-3y^2$.

8. Trace the curve $x^3+y^3=3axy$. Show that it has one double point and one rectilinear asymptote. Show also that the parabola $x^2=ay$ passes through this double point and the point where the ordinate of the curve is a maximum.

VIII.

1. Differentiate with respect to x the following expressions:

$$\frac{1}{2\sqrt{x}},\ \frac{3-5x}{x^2-x+1},\ \cos^3(2x),\ e^{ax}\sin bx,\ \log\tan\left(2x+\frac{\pi}{4}\right).$$

If y be a function of x, as given by the equation

$$xy+y+x=0,$$

explain the geometrical meaning of the differential coefficient of y with respect to x, when $x=4$. Hence write down the equation of the tangent to the curve at the point whose abscissa is 4.

2. Find the radius of the base of the cylinder of maximum volume, which can be cut out of a sphere of radius R. Find also the radius of the base, when the area of the curved surface of the cylinder is a maximum.

3. When the value of the differential coefficient of $f(x)$ is positive, what do you infer about the value of $f(x)$? If

$$y=(x-3)^5+3(x-4)^5+(x-8)^5,$$

prove that y vanishes for one, but not more than one, real value of x.

Show also that if x be positive, and less than unity, $\frac{1+x}{1-x}$ is greater than e^{2x}.

4. Find the equation of the tangent at the point (X, Y) to the curve $f(x, y)=0$.

Show that if the line $lx+my=1$ touches the curve

$$\left(\frac{x}{a}\right)^n+\left(\frac{y}{b}\right)^n=1,$$

then $$(al)^{\frac{n}{n-1}}+(bm)^{\frac{n}{n-1}}=1.$$

5. Find the radius of curvature at any point on the curve

$$Ax^2+By^2=1.$$

If the confocal conics

$$3x^2+8y^2=24,\ x^2-4y^2=4$$

have P as one of their points of intersection, find the radius of curvature of each of the curves at P, and show that the distance between the two centres of curvature is the square root of the sum of the squares of the radii of curvature.

6. What operations are represented by the expression

$$\frac{\partial^2 z}{\partial x\partial y},$$

where z is a function of x and y?

Prove that $$\frac{\partial^2 z}{\partial x \partial y}=\frac{\partial^2 z}{\partial y \partial x}.$$

Illustrate this when $$z=\log\tan\left(\frac{y}{x}\right).$$

Having given that x, y, z are connected by the two equations

$$x^2+y^2+z^2=a^2, \quad xyz=c^3,$$

find $$\frac{dx}{dz}, \ \frac{dy}{dz}.$$

7. If u be a homogeneous function of degree n in two independer variables x, y, show that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu,$$

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x \partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n(n-1)u.$$

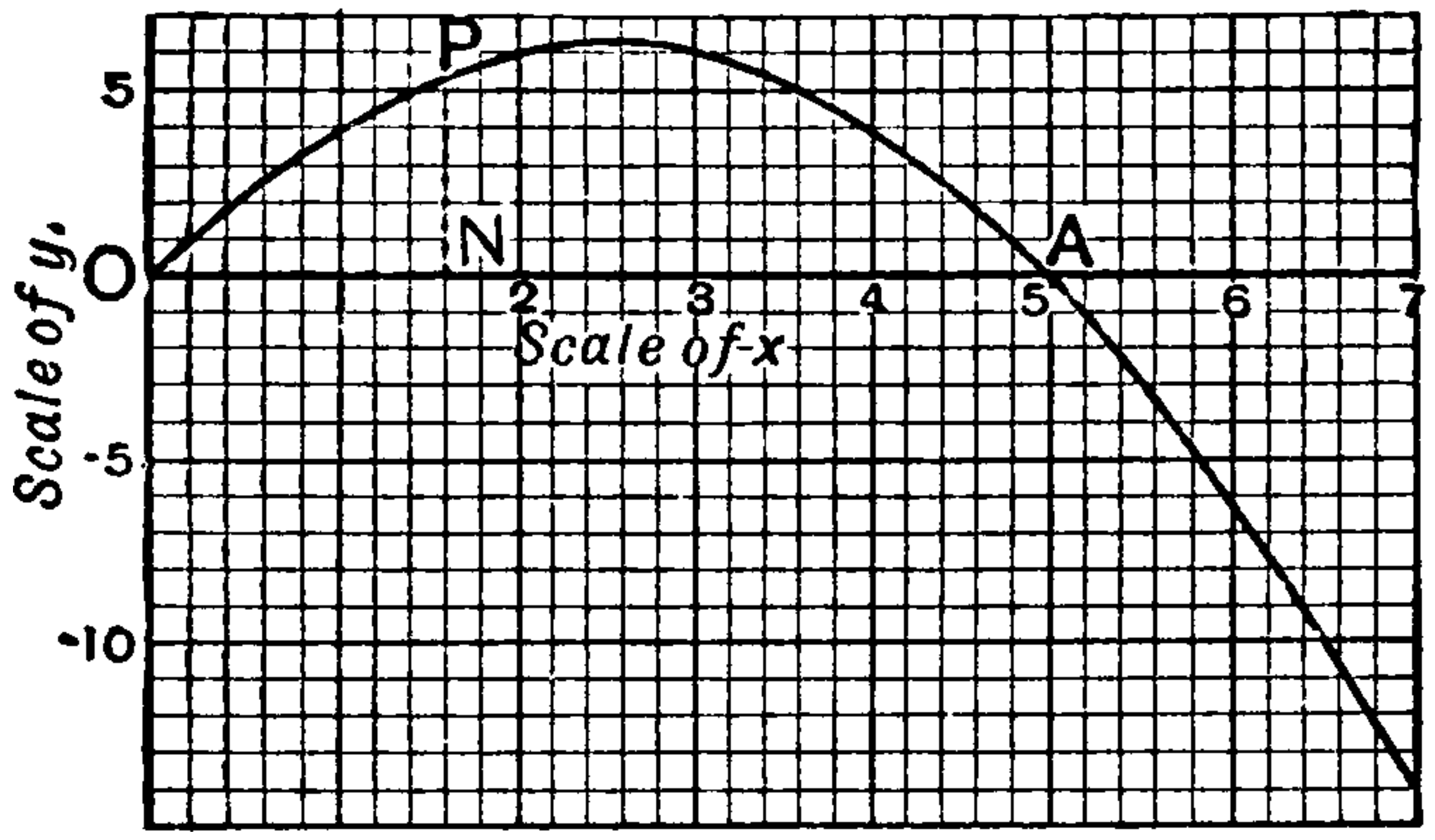
Verify the above by direct differentiation in the case when

$$u=(x^3+y^3)^{\frac{1}{2}}.$$

ANSWERS TO EXAMPLES.

Pages 15, 16.

No. 1.

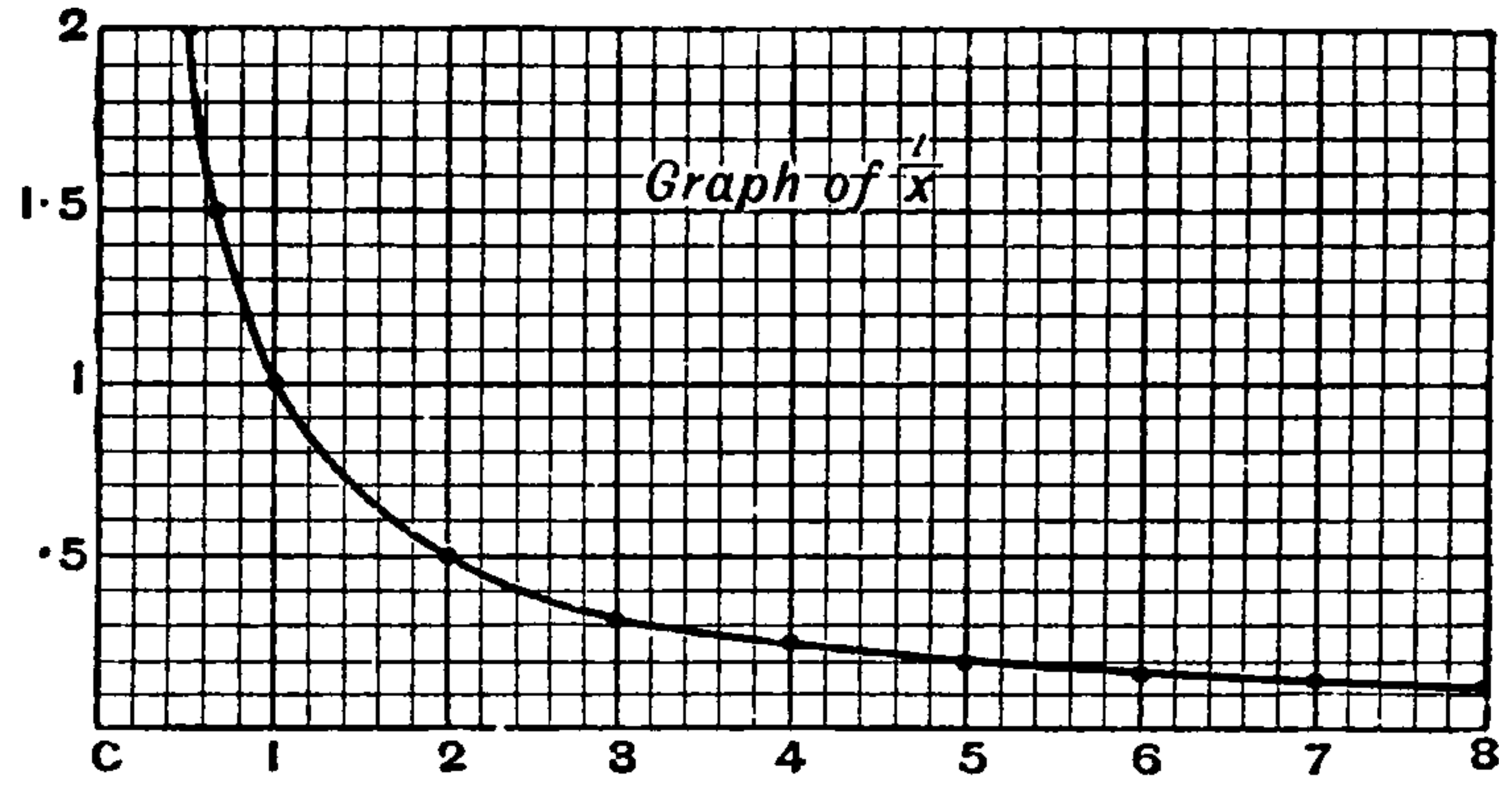


Denoting one of the numbers by x, the product is $y=x(5-x)$. The table of values is

x	0,	1,	2,	3,	4,	5,	6,	7, ...
y	0,	4,	6,	6,	4,	0,	-6,	-14, ...

from which the graph can be drawn, as shown. The greatest value of the product occurs when $x=2\frac{1}{2}$, *i.e.* when the two factors are equal.

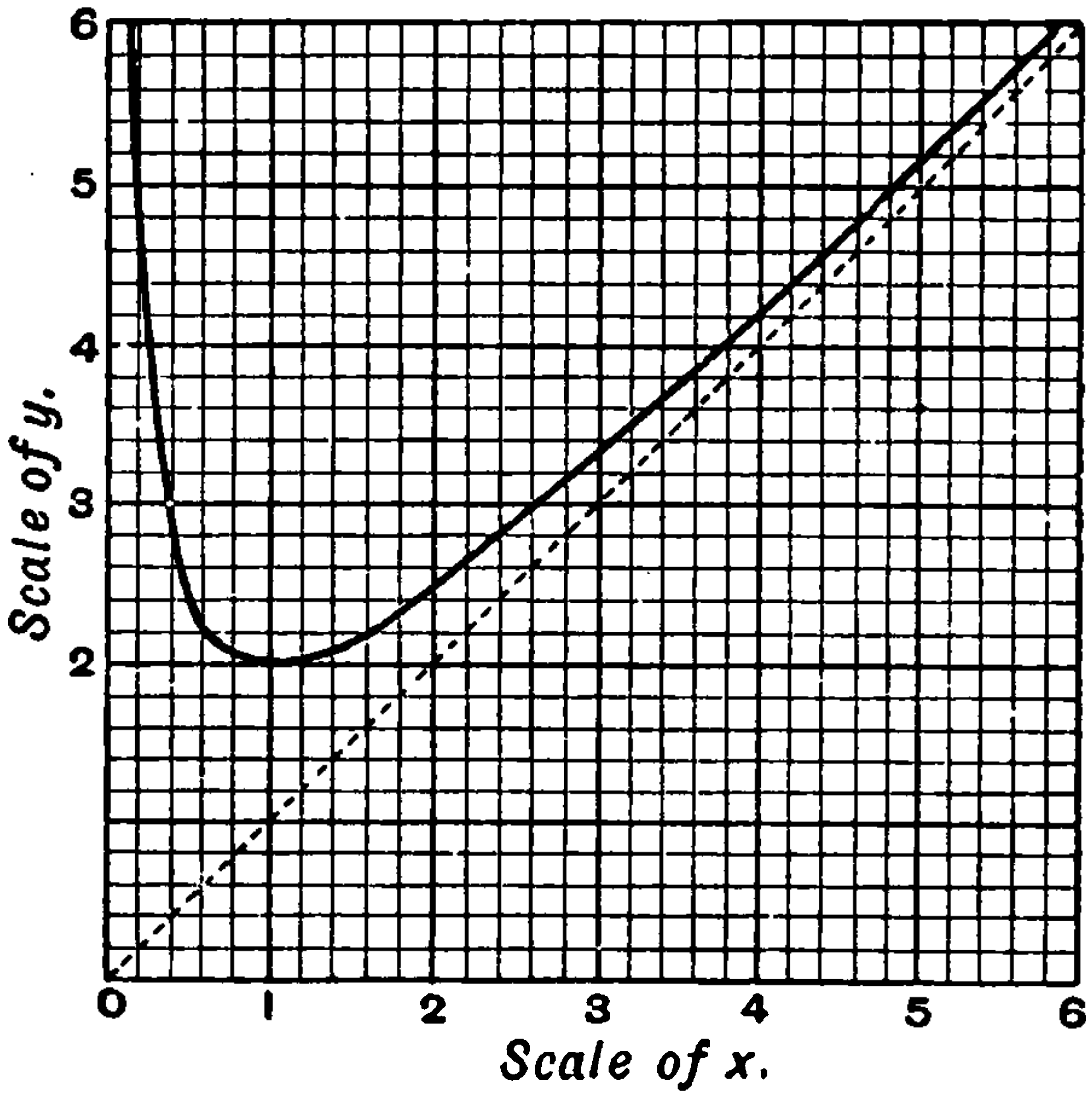
No. 2.



If $y=\dfrac{1}{x}$, the table of values, from which the graph can be drawn, is

x	0,	·5,	1,	2,	3,	4,	5, ...
y	∞,	2,	1,	·5,	·3,	·25,	·2, ...

No. 3.



The curve is $y=x+\frac{1}{x}$.

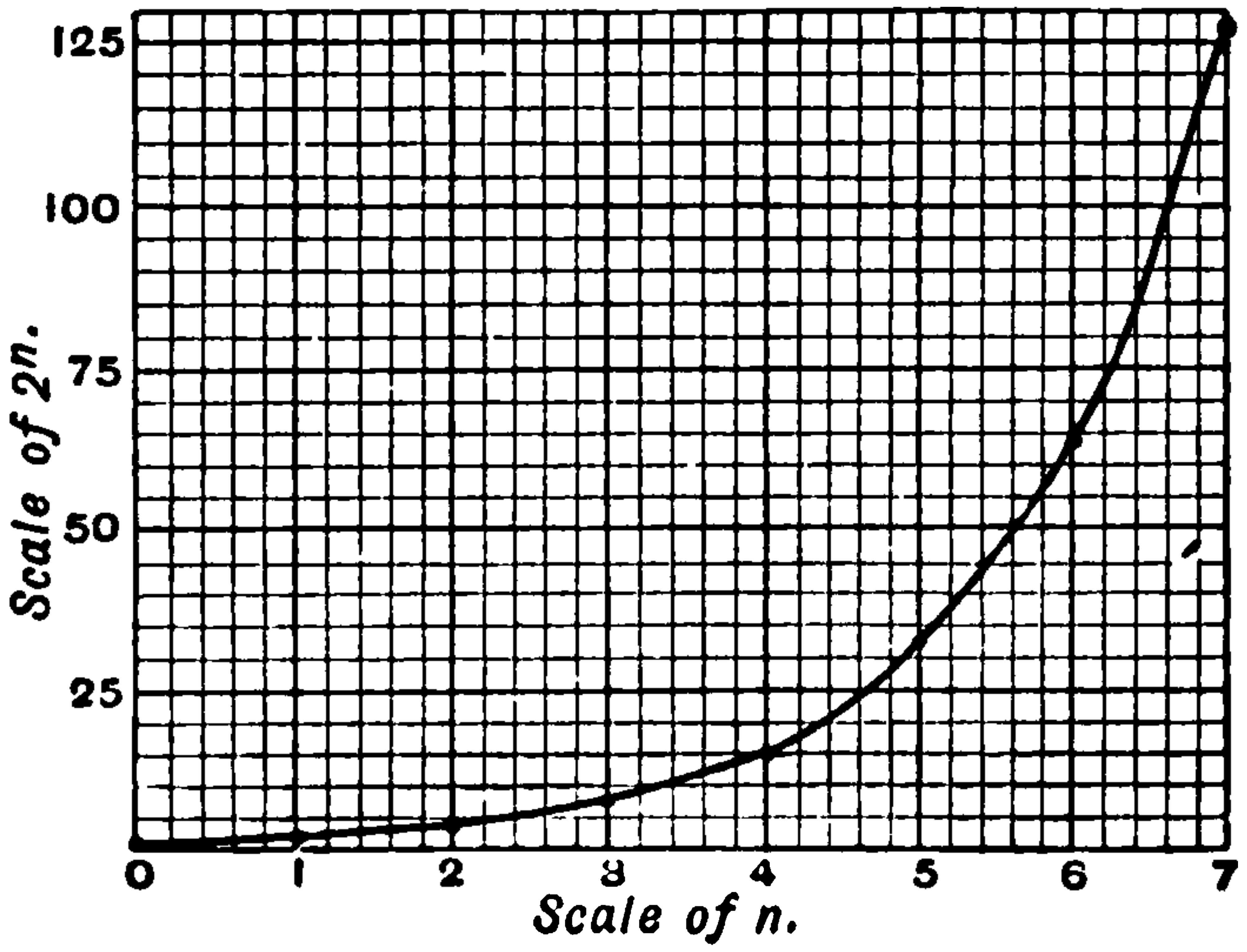
When x is large, the curve is only a very little above the line $y=x$, since the term $\frac{1}{x}$ becomes insignificant.

x	y
0	∞
1	2
2	2·5
3	3·33
4	4·25
5	5·2

The line $y=x$ is called an asymptote. It is shown by the dotted line in the diagram.

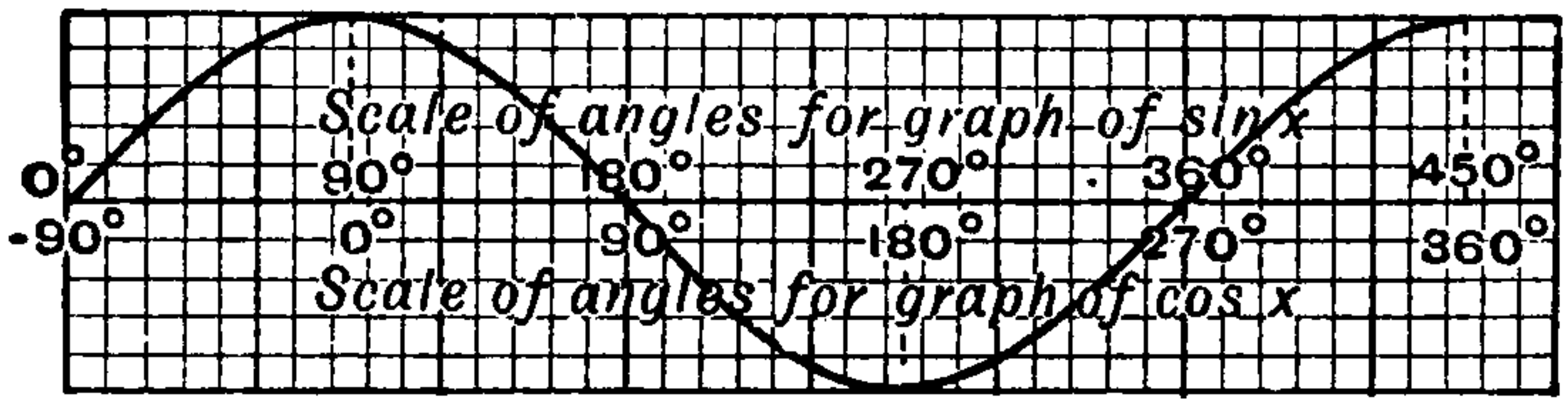
The curve is a hyperbola. Its other asymptote is the line $x=0$ (*i.e.* the axis of y). The minimum ordinate occurs when $x=1$.

No. 4.



If values of n are taken as abscissae, and values of 2^n as ordinates, the curve is roughly as shown. The vertical scale has to be very small, as the ordinates grow rapidly.

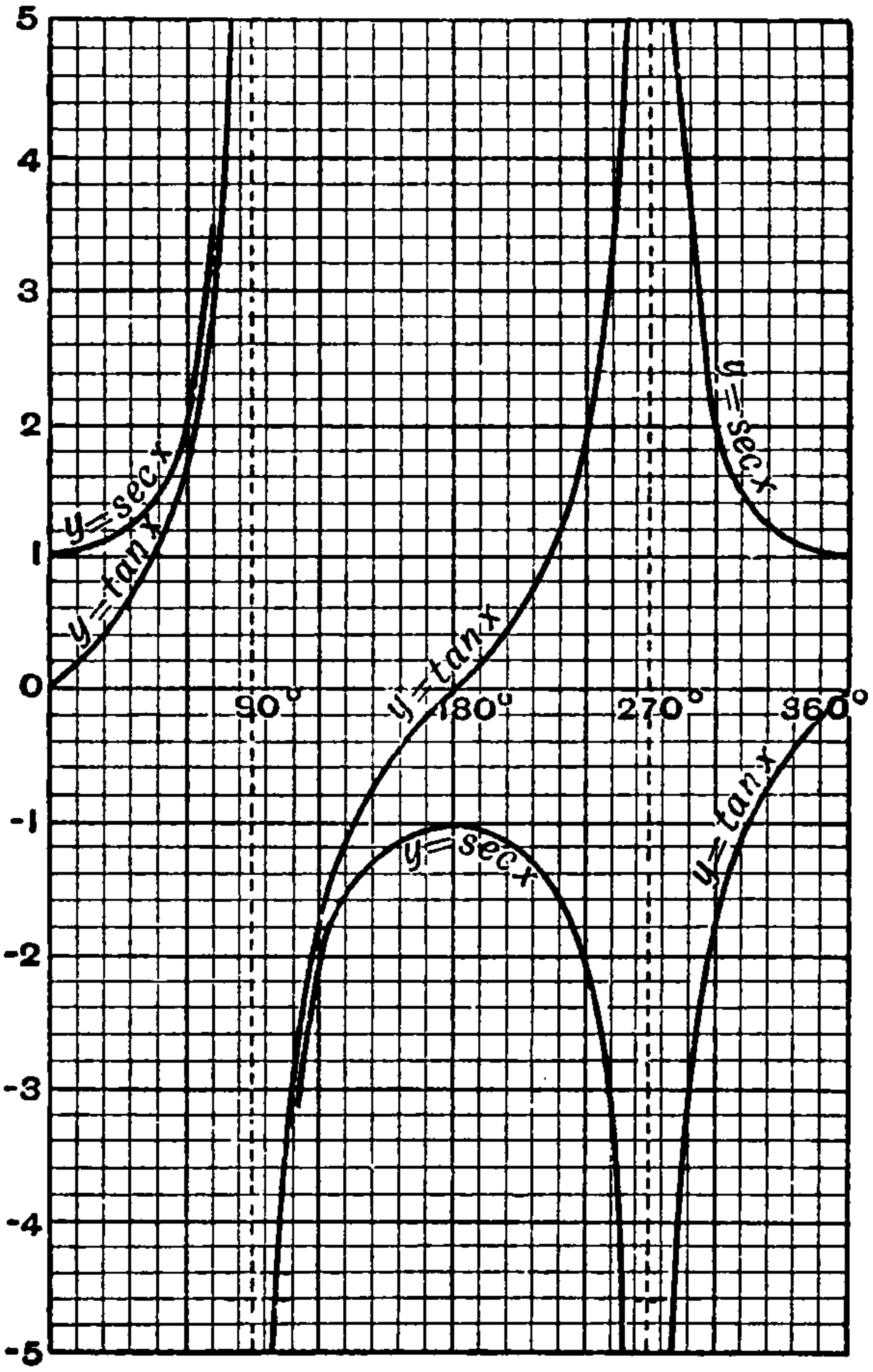
Nos. 5. and 6.



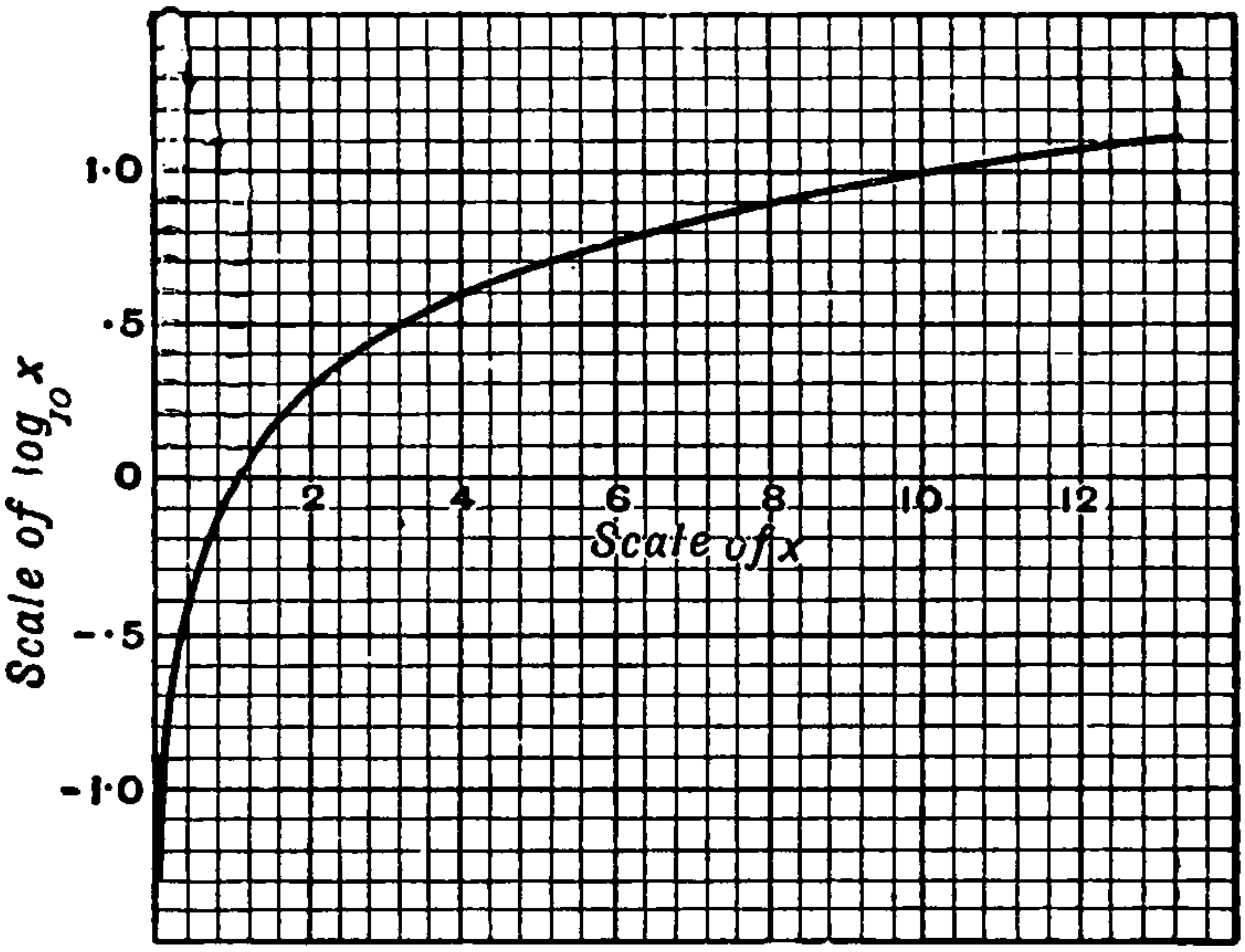
N.B. $\cos x = \sin(x+90°)$

Each horizontal division represents 12°. The student should draw this curve on a larger scale.

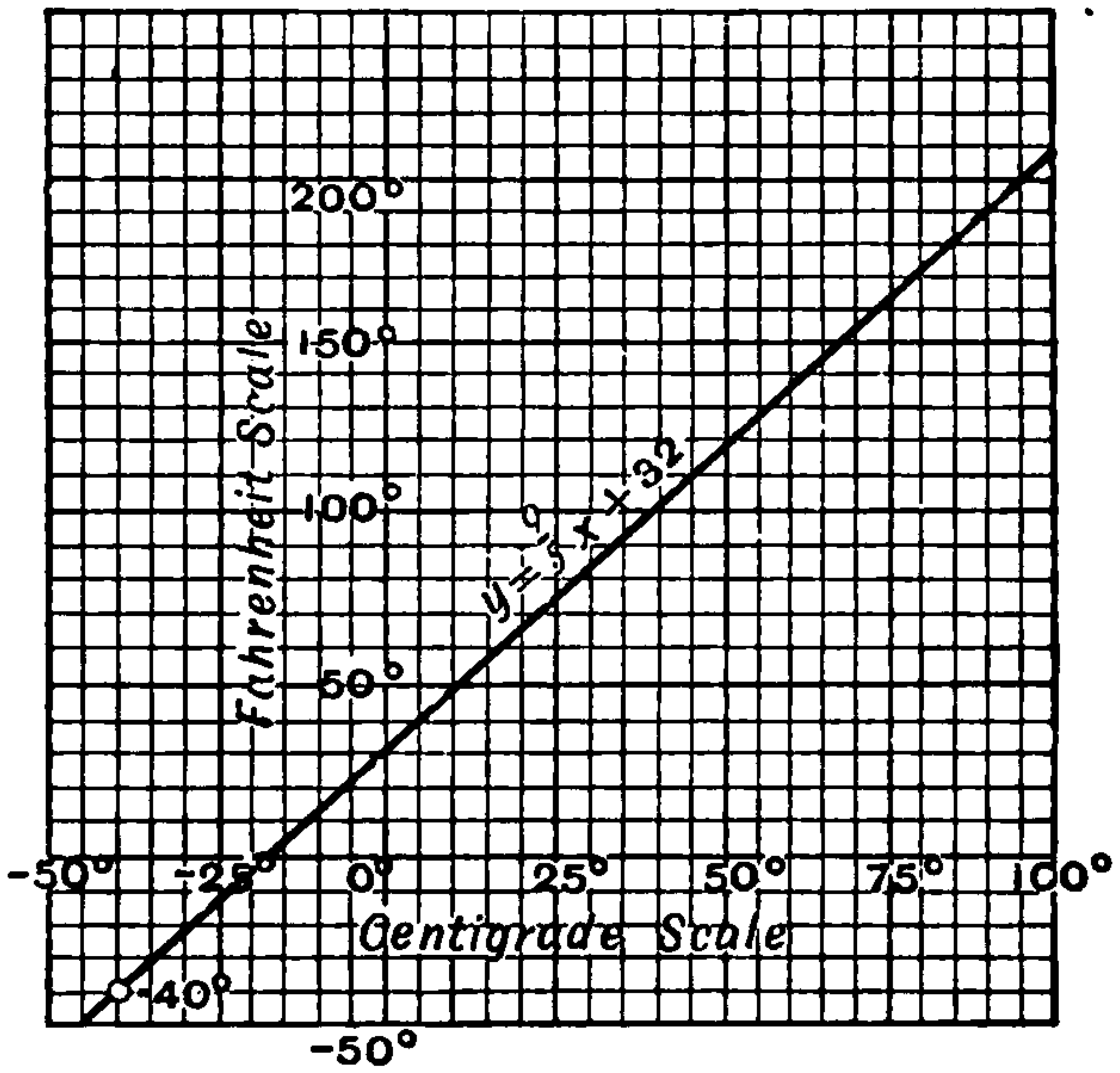
No. 7.



No. 8.



No. 9.

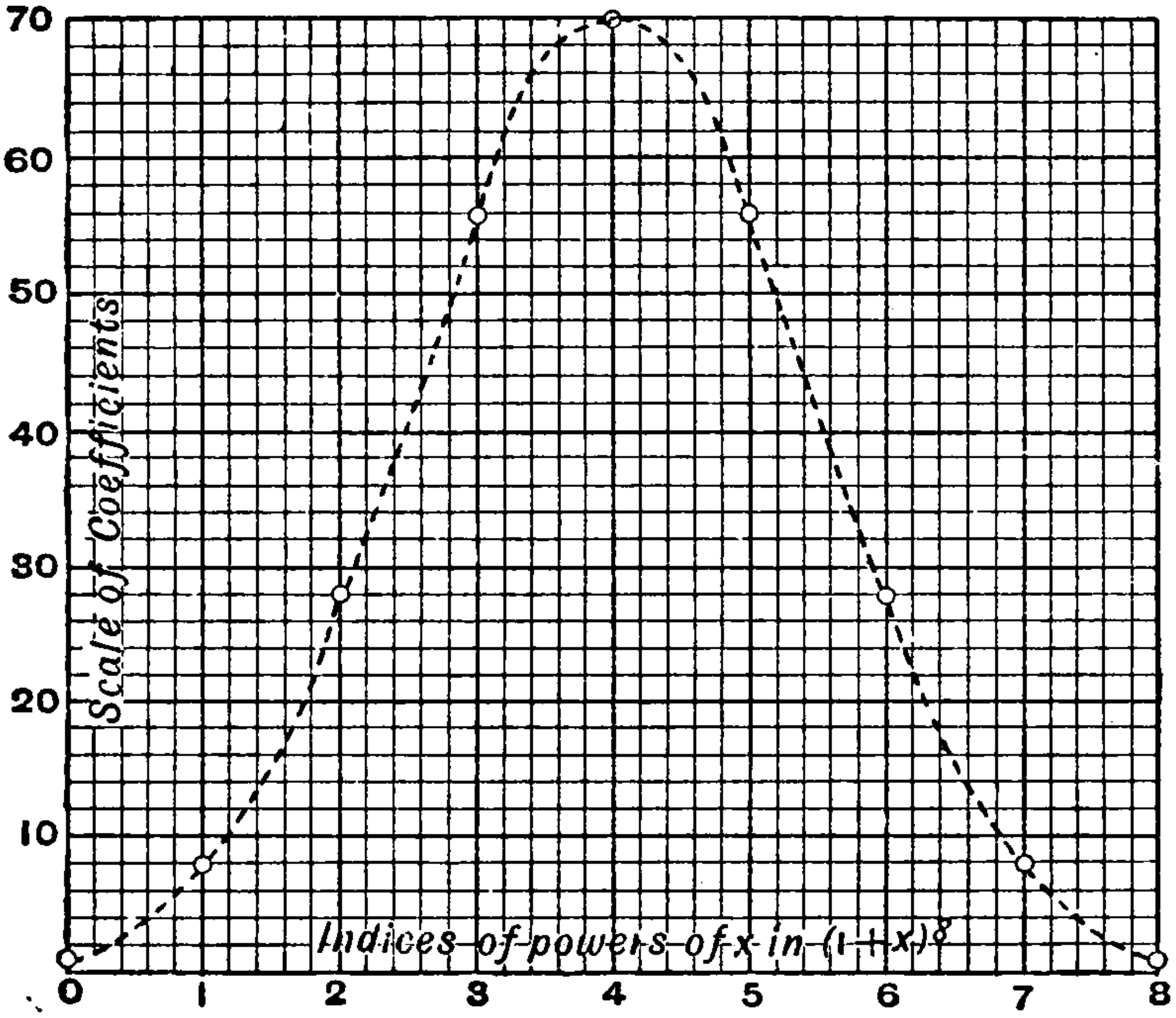


If x denotes any temperature in centigrade degrees, and y denotes the same temperature in Fahrenheit degrees, $y = 32$ when $x = 0$, and $y = 212°$ when $x = 100$. The graph is a straight line joining these points; its equation is

$$y - 32 = \frac{9}{5}x.$$

The readings on the two scales are equal for a temperature of $-40°$.

No. 10.



The required graph consists of a series of points corresponding to the integer values of x.

No. 11. The graph is a straight line going through the following points:

x, measured in feet	0,	1,	2,	3,	4,	5	etc.
y, measured in foot-lbs.	0,	10,	20,	30,	40,	50	

If the y-scale is $\frac{1}{10}$ of the x-scale, the line will be at 45° to the axis of x. It passes through O.

No. 12. If we measure the stretch in feet along the axis of x, and y in lbs. perpendicular to this axis, then when $x=0$, $y=0$, and when $x=6$, $y=10$. The graph is a straight line joining these points. Its equation is $y=\frac{5}{3}x$.

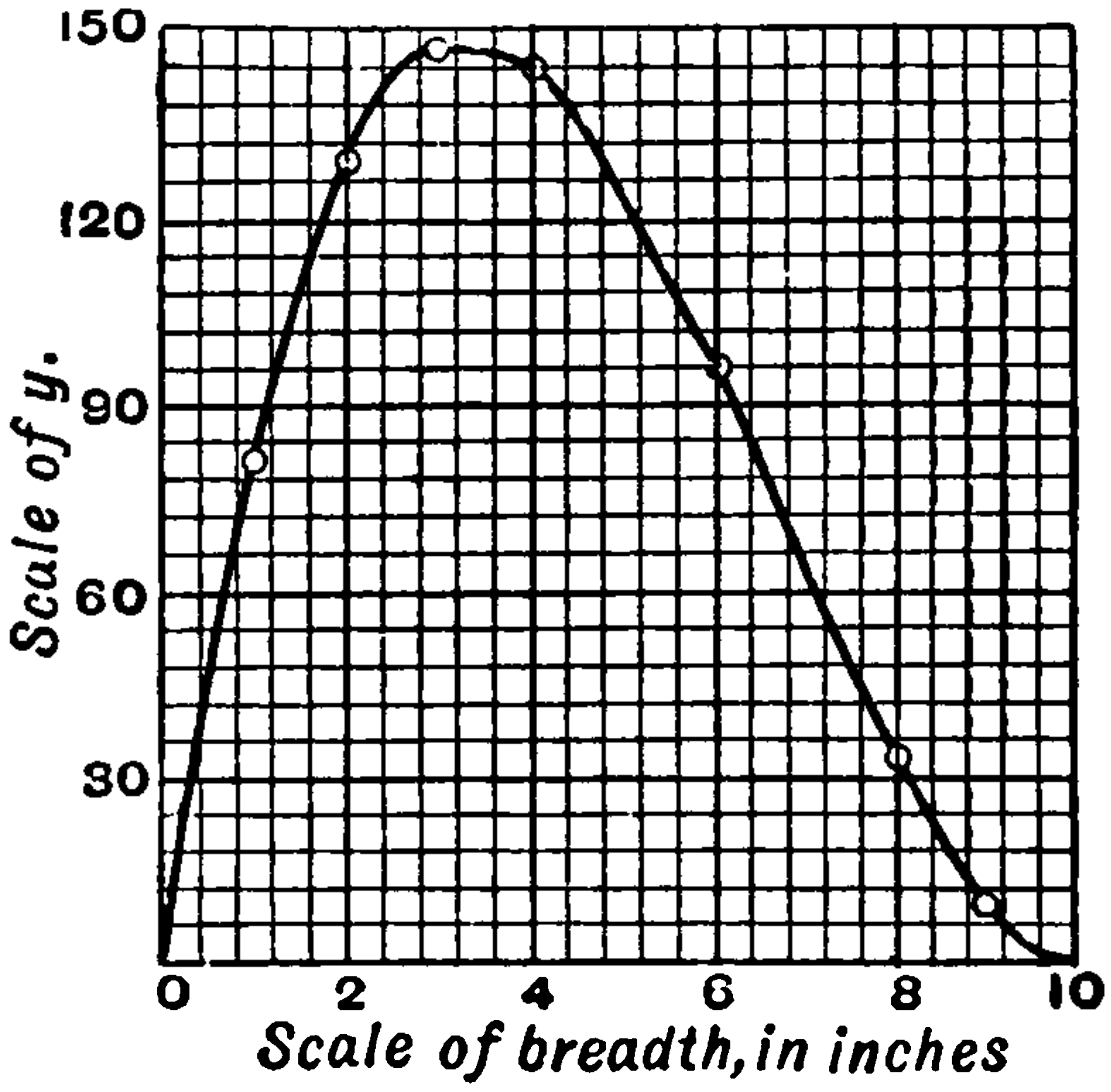
No. 13. Since the force increases uniformly with the amount of stretch, the average force throughout any given stretch x is half the final force, and is therefore $\frac{5}{6}x$. Hence the work done $=\frac{5}{6}x^2$, being the product of the average force into the stretch. This is the area of the triangle contained between the graph of the force, the axis of x, and the final ordinate corresponding to the value of x considered. The area will be so many foot-lbs., the base representing feet, and the height representing lbs.

If the work done in causing any stretch x is represented by an ordinate, its graph will be a parabola, and presents no difficulty. Its equation is $y=\frac{5}{6}x^2$.

No. 14. Since bd is constant, where b is the breadth and d the depth, and the strength is proportional to bd^2, it follows, on eliminating d, that the strength is inversely proportional to b. Therefore the graph of $\frac{1}{b}$ will show the changes in strength. This curve is shown in the answer to No. 2. The theorem does not hold for very thin beams unless they can be prevented from lateral buckling.

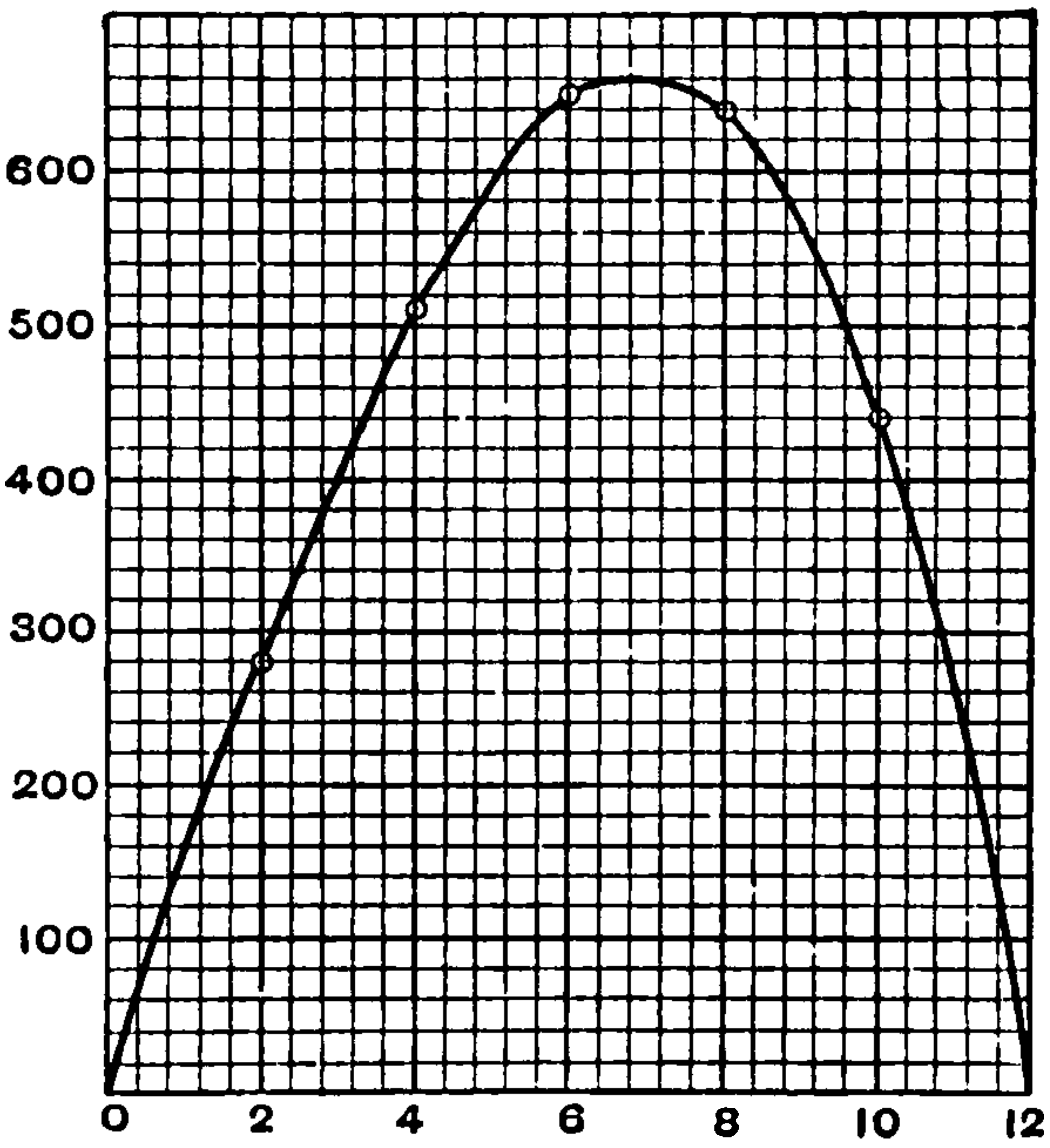
No. 15. The half-perimeter $=b+d=10$, and the strength is proportional to bd^2, which $=b(10-b)^2$. The graph is therefore $y=x(10-x)^2$ where x represents the breadth measured in inches.

x	y
0	0
2	128
4	144
6	96
8	32
10	0



No. 16. $y=x(144-x^2)$.

x	y
0	0
2	280
4	512
6	648
8	640
10	440
12	0



No. 17. From the diagram we see that the ordinate is a maximum when x is about equal to 7. A more accurate answer can be obtained

by equating the gradient to 0, as evidently the curve has zero gradient at the required point. By next chapter, or by Art. 18, we find the gradient is given by the values of $144-3x^2$, and this is equal to 0 when $x^2=48$, *i.e.* when x is just less than 7. The change in length of ordinate is so slight about here, however, that there will be no appreciable error in taking x *equal* to 7 for the maximum value of y, *i.e.* the strength is greatest when the breadth is about 7 inches and the depth $=\sqrt{95}$ or nearly 10 inches.

No. 18. From the diagram of No. 15, we see that y is a maximum when x is a little greater than 3·2 inches, which would make the depth a little less than 6·8 inches. So you might guess $b=3\frac{1}{3}$, $d=6\frac{2}{3}$, *i.e.* the depth double the breadth. If by help of next chapter, or Art. 18, we equate the gradient to 0, we shall find that this guess is correct.

Page 18.

1. $-\dfrac{x}{y}$. **2.** $-\dfrac{2x+A}{2y+B}$. **3.** $\dfrac{2a}{y}\,(=y/2x)$. **4.** $-\dfrac{b^2x}{a^2y}$.

5. $-\dfrac{a^2}{x^2}\,(=-y/x)$. **6.** $3x^2\left(=y\div\dfrac{1}{3}x\right)$. **7.** $\dfrac{3x^2}{2y}\left(=y\div\dfrac{2}{3}x\right)$.

Pages 22-24.

1. $3x^2-40x+100$; zero gradient when $x=3\frac{1}{3}$ and 10.

2. $144-3x^2$; zero gradient when $x=\pm\sqrt{48}$.

3. $\dfrac{3}{2\sqrt{x}}-\dfrac{15\sqrt{x}}{2}-\dfrac{1}{x\sqrt{x}}$. **7.** $\dfrac{3}{2}\sqrt{x}$.

10. $4x(x-1)(x-2)$.

x	-1	0	1	2	3
$f(x)$	9	0	1	0	9
$f'(x)$	-24	0	0	0	24

11. $-x^{-2}+x^{-3}-x^{-4}$. **12.** $-\dfrac{4x}{9y}$.

13. The graph is a parabola whose highest point is at $(1\frac{9}{16}, 139\frac{1}{16})$.

14. Velocity $=0$ when $t=1\frac{9}{16}$. The velocity graph is a straight line whose intercepts on the axes are $1\frac{9}{16}$, 50.

15. (1) $-2(6x-5)(3x^2-5x+6)^{-3}$; (2) $-\dfrac{4x}{(x^2-6)^2}+\dfrac{4x}{(x^2+6)^2}$;

(3) $\dfrac{2x-3}{2\sqrt{(x^2-3x+2)}}$; (4) $1-\dfrac{x}{\sqrt{(x^2-1)}}$; (5) $-\dfrac{5}{3}\left(\dfrac{y}{x}\right)^{\frac{1}{3}}$.

16. $\dfrac{2}{3}$. **17.** Gradient in each case $=2Ap+B$.

Page 30.

1. (1) $\dfrac{-2a}{(x-a)^2}$; (2) $\dfrac{3x-a}{2\surd(x-a)}$; (3) $\dfrac{-a}{(x+a)^{\frac{1}{2}}(x-a)^{\frac{3}{2}}}$;

(4) $10x+28x^3+18x^5$; (5) $\dfrac{6x(1-x)(1+x^2)^2}{(1+x^3)^3}$; (6) $\dfrac{a^2}{(a^2-x^2)^{\frac{3}{2}}}$.

3. $-\dfrac{1}{(a+x)^2}$. **4.** $\dfrac{2}{(a+x)^3}$; $\dfrac{-2.3}{(a+x)^4}$; $\dfrac{2.3.4}{(a+x)^5}$; $\dfrac{-2.3.4.5}{(a+x)^6}$

5. 588·88. **6.** 0.

7. $\dfrac{x^4-2a^2x^2-a^4}{(x^2-a^2)\surd(x^4-a^4)}$. **8.** $(x-3)(x+1)^2(12x^2-53x+43)$.

9. $\dfrac{x(a^2-y^2)}{y(a^2+x^2)}$. **10.** $f'(z) \,.\, \phi'(x)$.

11. $24(x-1)(3z-1)\surd(3z^2-2z+5)$. **12.** $\dfrac{9(z^2+5)(x^2+5)}{(z^2-5)^2(x^2-5)^2}$.

Pages 44, 45.

1. 0. **2.** $-2\sin 2x$.

3. $3\sec^2 3x+4\operatorname{cosec}^2 4x$. **4.** $-\dfrac{\cos 3x}{2\cos^5 x}$.

5. $\sec x(\sec x+\tan x)$. **6.** $\cos 2x=\cos^2 x-\sin^2 x$.

7. $\sin 3x=4\cos^2 x\sin x-\sin x=3\sin x-4\sin^3 x$.

8. $(4\sec^4 x-12\sec^3 x+4\sec^2 x)\tan x$. **9.** $15\sec^6 x$.

10. $\dfrac{4x}{\surd(1-4x^4)}$. **11.** $\sin^{-1}x$. **12.** $2x\tan^{-1}x$.

13. $\dfrac{1}{\surd(1-x^2)}$. **14.** $-\dfrac{1}{2(x^2+1)}$. **15.** $\dfrac{2}{x^2+1}$.

16. $y=0$ when $x=0, 2, 4, \ldots$; and at these points $\dfrac{dy}{dx}=\pm\dfrac{\pi x}{4}$, alternately. $\dfrac{dy}{dx}=0$ when $\tan\dfrac{\pi x}{2}=-\dfrac{\pi x}{2}$. The curve is symmetrical on either side of the axis of y; *i.e.* the axis of y is a line of symmetry.

17. $y=\pi$ when $x=0$ (applying Art. 32). $\dfrac{dy}{dx}=0$ when $\tan\dfrac{\pi x}{2}=\dfrac{\pi x}{2}$. $y=0$ when $x=2, 4, 6, \ldots$; and at these points $\dfrac{dy}{dx}=\mp\dfrac{\pi}{x}$.

The curve cuts the axis of y at right angles (the proof of this is not easy till after we know how to expand $\sin\dfrac{\pi x}{2}$ and $\cos\dfrac{\pi x}{2}$ in powers of x), and this axis is a line of symmetry.

18. (1) Similar to No. 16, but on half-scale.

(2) The ordinates are double, and the abscissae are half, of those which give the corresponding points in No. 17.

19. $y = \pm\frac{2}{x}$ when $x = 0, 2, 4, 6, \ldots$. $\frac{dy}{dx} = 0$ when $\cot\frac{\pi x}{2} = -\frac{\pi x}{2}$.

$y = 0$ when $x = 1, 3, 5, \ldots$; and at these points $\frac{dy}{dx} = \mp\frac{\pi}{x}$.

The left-hand portion is an inverted copy of the right-hand portion, so that the origin is a centre of symmetry.

20. $y = \pm\frac{1}{2}x$ when $x = 0, 2, 4, 6, \ldots$. $\frac{dy}{dx} = 0$ when $\cot\frac{\pi x}{2} = \frac{\pi x}{2}$.

$y = 0$ when $x = 1, 3, 5, \ldots$; and at these points $\frac{dy}{dx} = \mp\frac{\pi x}{4}$.

The gradient at the origin is $\frac{1}{2}$. The origin is a centre of symmetry.

Pages 49, 50.

1. $x + \frac{1}{3}x^3 + \frac{2}{15}x^5$.

2. $1 + \frac{1}{2}x^2 + \frac{5}{24}x^4$.

4. $1 - 2x^2 + \frac{2}{3}x^4 - \frac{4}{45}x^6 + \frac{2}{315}x^8 - \ldots$.

5. $\cos^2 x = 1 - x^2 + \frac{1}{3}x^4 - \frac{2}{45}x^6 + \frac{1}{315}x^8 - \ldots$.

6, 7. Calculate the ordinates and gradients for $x = 1, \sqrt{2}, 2, \sqrt{6}, 3$.

The corresponding values of $\sin x$ are ·841, ·988, ·909, ·638, ·141; and those of $\cos x$ are ·540, ·156, − ·416, − ·770, − ·990.

It will be found that the respective series graphs approximate to the above more closely as more terms are taken, but that each correction is rather too great.

Pages 62, 63.

1. $\cot x$.

2. $\frac{1+x}{1+x^2}$.

3. $\frac{x+3}{(x+1)^2}$.

4. $2xy$.

5. $1 + \log x$.

6. $\sec x$.

7. $\frac{2(1-x^2)}{1+x^2+x^4}$.

8. $y \cos x \,.\, \log_e a$.

9. $y(x\cos x + \sin x)$.

10. $\frac{y(x\cos x - \sin x)}{x^2}$.

11. $y(x^{-1} - 2x)$.

12. $y \sin x(\cot^2 x - \log\sin x)$.

13. $2\,\mathrm{tau}^3 x$.

14. $x^{n-1}(1 + n\log x)$.

15. $\frac{1}{x\log x}$.

16. $\frac{e^x - e^{-x}}{e^x + e^{-x}}$.

22. (1) $-x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \ldots$; (2) $2\left(x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \ldots\right)$.

23. $\log_e 2 = 0{\cdot}693147\ldots$.

Pages 74, 75.

3. Population in x years $=10^6 \,.\, 2^{\frac{x}{100}}$.
Rate of growth per 1000 per annum $=6{\cdot}93(+)$.
1,148,700 ; 1,414,200 ; 1,741,100.

4. 50,000 lbs. weight.

5. $a_1=10$; $a_2=1{\cdot}443$, or about $\frac{13}{9}$ or $\frac{88}{61}$;
$a_3=1{\cdot}0064$, or about $\frac{157}{156}$.

6. (1) $3{\cdot}16\ldots$; (2) $1{\cdot}778\ldots$.

7. $k={\cdot}366\ldots$. [Students of dynamics will know that this is the coefficient of friction between the cable and post.]

8. 14·8 feet.

14. A little less than 36 per cent. (assuming there is no selective absorption).

15. 90·44 per cent.

16. $1\frac{2}{5}$ seconds (nearly).

17. About 22 minutes.

18. 5 minutes 28 seconds (nearly) ; $47\frac{1}{2}$ seconds (nearly). Curve of fall is $y=({\cdot}9)^{\frac{x}{5}}$ if x is measured in seconds; the subtangent is $47\frac{1}{2}$ seconds. The curve of rise is $1-({\cdot}9)^{\frac{x}{5}}$. It is an image of the curve of fall with respect to the line $y=\frac{1}{2}$. Both curves cross this line when $x=33$ seconds (nearly).

Page 83.

1. $2\sinh x \cos x$.

2. $2\cosh x \cos x$.

3. $\tanh x$.

4. $2\,\mathrm{sech}\, x$.

Miscellaneous Examples. Page 83.

1. $\dfrac{2x-y-2a}{x-4y+6a}$.

2. $\dfrac{2}{1-x^4}$.

3. $$\{1-(\log x)^2\}^{\frac{1}{2}} \,.\, e^x\left(\log x+\frac{1}{x}\right) \,.\, \cos(e^x\log x)$$
$$-\{1-(\log x)^2\}^{-\frac{1}{2}} \,.\, \frac{\log x}{x} \,.\, \sin(e^x\log x).$$

4. $-e^{-ax}(a\cos 3x+3\sin 3x)$.

5. $2x(\sin 2x+x\cos 2x)$.

6. $-\dfrac{2pq^2r^2(2p^2+q^2+r^2)}{(p^2+q^2)^2(p^2+r^2)^2}$; $\dfrac{2p^2qr^2}{(p^2+q^2)^2(p^2+r^2)}$; $\dfrac{2p^2q^2r}{(p^2+q^2)(p^2+r^2)^2}$.

7. 0.

8. $-a\tan\theta\sec\theta \div \surd(\cos 2\theta)$.

9. $(1-x)^{\frac{1}{2}}(1+x)^{-\frac{3}{2}}$.

10. $\cot\frac{1}{2}\theta$.

11. $\dfrac{-24x^2-42xz+64z^2}{(4x-7z)^5}$; $\dfrac{84x^2-32xz-56z^2}{(4x-7z)^5}$; $-2y$.

Page 94.

2. 1·325.
3. 1·4236.
4. The circular measure of 49° 17′ 36½″.
5. 1·37129.
6. 38° 10′ 21·7″.
8. ·000287656.
9. The roots are 0·75736 and 9·24264.
10. The roots are 10·6569 and −0·6569.
11. The roots are 0·7386 and −4·7386.

Page 109.

1. Height = 1 foot; volume $=\frac{4\pi}{9}$ cubic feet.
2. $h=r$.
3. $h=2r$.
4. (1) $h^2=2r^2$; (2) $h^2=8r^2$.
5. $h^2=\frac{1}{3}l^2$; $r^2=\frac{2}{3}l^2$.
6. $r^2=\frac{2}{3}a^2$, $h^2=\frac{4}{3}a^2$; volume $=\frac{1}{\sqrt{3}}$ of volume of sphere.
7. Width = twice depth.
8. 15 feet 6½ inches; 46° 44′ 24″.
9. Max., 12 (when $x=0$); min., −20 (when $x=4$).

10.

x	−2	−1	1	2
y	−16 max.	−38 min.	38 max.	16 min.

11. $2\sqrt{(ab)}$.
12. $(3, \frac{3}{4})$, $(0, 0)$, $(-3, -\frac{3}{4})$.
13. Max., 4 (when $x=0$); min., 0 (when $x=2$).
14. Max., 0 (when $x=-1$); min., −8·4 (when $x=\frac{1}{6}$).
    Point of inflexion at (2, 0).
15. Max., 18·36 (when $x=\frac{2}{7}$); min., 0 (when $x=2$).
    Point of inflexion at (−1, 0).

16. None. The curve is a hyperbola with asymptotes $x=y$ and $x=1$. The *conjugate* hyperbola has a maximum ordinate at (0, −1), and a minimum ordinate at (2, 3).

17. Max. value = 54; min. = −71.

Page 123.

1. If a, b are the lengths of the horizontal and vertical units of scale; $\rho=\frac{4a^2}{b}\sec^3\phi$, where $\tan\phi=-\frac{b}{4a}$.
   Hence, (1) $\rho=4{\cdot}37$; (2) $\rho=1{\cdot}4b=2{\cdot}8a$.
2. (1) $\rho=\sqrt{2}$; (2) $\rho=1{\cdot}4b=2{\cdot}8a$.
3. ½.

4. $\rho=\dfrac{(a^2+y^2)^{\frac{3}{2}}}{ay}$. **5.** $\rho=\dfrac{3\sqrt{3}}{2}a$; when $y^2=\dfrac{1}{2}a^2$.

6. $\rho=\dfrac{y^2}{a}=y\sec\phi$, since $\sec\phi=\dfrac{y}{a}$. **7.** $\dfrac{1}{20}$. **8.** $\dfrac{1}{12\sqrt{12}}$.

9. (1) $\dfrac{113\sqrt{226}}{2}$; (2) $\dfrac{25a\sqrt{10}}{2}$. **10.** $\rho=\dfrac{5a}{12}=\dfrac{25b}{12}$, at each point.

12. $(a^2\sin^2\theta+b^2\cos^2\theta)^{\frac{3}{2}}\div ab$. **13.** $(a^2\tan^2\theta+b^2\sec^2\theta)^{\frac{3}{2}}\div ab$.

Page 125.

1. 360. **2.** $-\dfrac{15}{16}x^{-\frac{3}{2}}-\dfrac{63}{16}x^{-\frac{5}{2}}-\dfrac{15}{2}x^{-\frac{7}{2}}$.

Pages 132, 133.

1. (1) $D^4y=24$; $D^{-4}y=\dfrac{x^8}{1680}-\dfrac{x^6}{120}+\dfrac{x^5}{60}-\dfrac{x^4}{24}+D^{-4}(0)$.

(2) $D^4y=\dfrac{1680}{x^9}-\dfrac{720}{x^7}+\dfrac{24}{x^5}$; $D^{-4}y=\dfrac{1}{24x}+\left(\dfrac{x^3}{6}-x\right)\log x+D^{-4}(0)$.

(3) $D^4y=\dfrac{1680}{(x+a)^9}$; $D^{-4}y=\dfrac{1}{24(x+a)}$.

(4) $D^4y=-\dfrac{15}{16}x^{-\frac{7}{2}}$; $D^{-4}y=\dfrac{16}{945}x^{\frac{9}{2}}$.

2. $\dfrac{(-1)^n\lfloor\underline{n}}{(x-a)^{n+1}}$ and $\dfrac{\lfloor\underline{n}}{(a-x)^{n+1}}$.

3. (1) $D^ny=(-1)^{n-1}\dfrac{1.3.5\ldots(2n-3)}{2^n}x^{\frac{1}{2}-n}$; $D^{-n}y=\dfrac{2^nx^{\frac{1}{2}+n}}{3.5\ldots(2n+1)}$.

(2) $D^ny=(-1)^n\dfrac{1.3.5\ldots(2n-1)}{2^n}x^{-\frac{2n+1}{2}}$; $D^{-n}y=\dfrac{2^nx^{n-\frac{1}{2}}}{1.3.5\ldots(2n-1)}$.

6. If the graph of $\lfloor\underline{n}$ is plotted by drawing a curve through the points corresponding to positive integer values of n, the ordinates corresponding to fractional values of n may be taken as the corresponding values of $\lfloor\underline{n}$.

The second part of the question is solved by using the formula

$$\lfloor\underline{n}=n\lfloor\underline{n-1}.$$

Thus $$\frac{\lfloor\underline{\tfrac{1}{2}}}{\lfloor\underline{\tfrac{1}{2}-4}}=\tfrac{1}{2}(\tfrac{1}{2}-1)(\tfrac{1}{2}-2)(\tfrac{1}{2}-3)=-\tfrac{15}{16}.$$

7. (1) $\dfrac{x^{n-r}}{\lfloor\underline{n-r}}$, $\dfrac{x^{n+r}}{\lfloor\underline{n+r}}$; (2) 1, $\dfrac{x^{2r}}{\lfloor\underline{2r}}$. **9.** $\lfloor\underline{n}\left\{\log x+\sigma\left(\dfrac{1}{n}\right)\right\}$.

Page 136.

1. $5^r e^{4x} \sin\left(3x + r\tan^{-1}\frac{3}{4}\right)$.

3. $(-1)^r(a^2+b^2)^{\frac{r}{2}} e^{-ax} \sin\left(bx - r\tan^{-1}\frac{b}{a}\right)$.

$$D^{-1}y = -\frac{e^{-ax}}{\sqrt{(a^2+b^2)}}\sin\left(bx+\tan^{-1}\frac{b}{a}\right) = -\frac{e^{-ax}(a\sin bx + b\cos bx)}{a^2+b^2}.$$

4. Deducible from (3), by writing $\cos bx = \sin\left(bx+\frac{\pi}{2}\right)$.

$D^{-1}y = e^{-ax}(b\sin bx - a\cos bx) \div (a^2+b^2)$.

Page 143.

2. $\frac{(-1)^{n-2}\underline{|n-2}}{x^{n-1}}$; $\frac{2(-1)^{n-3}\underline{|n-3}}{x^{n-2}}$. Proved either by Leibnitz, or by direct differentiation.

3. $xD^{-3}\log x - 3D^{-4}\log x = \text{etc.} = \frac{x^4}{\underline{|4}}\left(\log x - \frac{1}{2} - \frac{1}{3} - \frac{1}{4}\right)$.

4. $-27e^{-3x}\{9x^2 - 57x + 101\}$; $-\frac{1}{27}e^{-3x}\left\{x^2 - x + \frac{7}{3}\right\}$.

5. $(32x^2-160)\cos 2x + 160x\sin 2x$; $\frac{1}{8}\{(x^2-3)\cos 2x - 3x\sin 2x\}$.

7. $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} = (n^2-m^2)y_n$.

9. $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} = (n^2+a^2)y_n$.

Page 145.

(1) $1 - \frac{2x^2-3x}{x^3-1}$; (2) $\frac{1}{2}x + \frac{7x}{4x^3-14}$;

(3) $\frac{1}{25}\left\{5x+7+\frac{29x+21}{5x^2-2x-3}\right\}$.

Page 146.

1. (1) $\frac{(-a)^n\underline{|r+n-1}}{\underline{|r-1}(ax+b)^{r+n}}$; $\frac{-1}{a(r-1)(ax+b)^{r-1}}$.

(2) The same as (1) with the sign of a changed.

2. $\frac{1}{2}\log(2x-3)$; $\frac{-3}{4(2x-3)^2}$; $-\frac{2}{5}\log(2-5x)$; $\frac{4}{5(2-5x)}$.

3. The two last.

Page 154.

1. $y=\dfrac{1}{x-2}+\dfrac{2}{x+3}$; $\quad y_{-1}=\log(x-2)+2\log(x+3)$.

2. $y_{-1}=3\log(x-2)-\log(x+1)$.

3. $y_{-1}=\dfrac{1}{x+1}+2\log(x+2)-\log(x+1)$.

4. $y_{-1}=\log(x-3)-\dfrac{1}{3}\log x-\dfrac{2}{3}\log(x+3)$.

5. $y_{-1}=(c-b)\log(x-a)+(a-c)\log(x-b)+(b-a)\log(x-c)$.

6. $y_{-1}=\dfrac{1}{2}\log(2x-3)-\dfrac{3}{2x-3}-\dfrac{9}{4(2x-3)^2}$.

7. $y_{-1}=\dfrac{1}{3}x^3+\dfrac{1}{2}x^2+13x+\dfrac{1}{7}\{251\log(x-4)-97\log(x+3)\}$.

8. $y_{-1}=\dfrac{37}{125}\log\dfrac{x-3}{x+2}-\dfrac{2}{25}\left(\dfrac{7}{x-3}+\dfrac{12}{x+2}\right)$.

9. $y_{-1}=\dfrac{2}{3}\log\dfrac{x-1}{x+2}-\dfrac{1}{2(x-1)^2}$.

10. $y=\dfrac{1}{3(x-2)^3}+\dfrac{7}{27(x-2)^2}-\dfrac{44}{243(x-2)}+\dfrac{44x+69}{243(x^2+x+3)}$.

11. $y_{-1}=-\dfrac{7}{6(x-2)^2}-\dfrac{43}{18(x-2)}+\dfrac{89}{216}\log(x-2)$
$\qquad\qquad -\dfrac{1}{27}\log(x+1)-\dfrac{3}{8}\log(x+2)$.

12. $y=\dfrac{4}{2x+3}+\dfrac{3-2x}{x^2+1}$; $\quad y_{-1}=2\log(2x+3)+3\tan^{-1}x-\log(x^2+1)$.
(This integral can be verified by differentiation.)

Page 161.

1. $\dfrac{1}{2}\left\{\dfrac{x-3}{x^2+x+1}+\dfrac{5x-1}{x^2-x+1}\right\}$.

2. $\dfrac{1}{2\sqrt{2}}\left\{\dfrac{x+\sqrt{2}}{x^2+x\sqrt{2}+1}-\dfrac{x-\sqrt{2}}{x^2-x\sqrt{2}+1}\right\}$.

3. $\dfrac{1}{2\sqrt{3}}\left\{\dfrac{x+2\sqrt{3}}{x^2-x\sqrt{3}+1}-\dfrac{x-2\sqrt{3}}{x^2+x\sqrt{3}+1}\right\}$.

4. $x+\dfrac{1}{3}\left\{\dfrac{1}{x+1}-\dfrac{x+1}{x^2-x+1}\right\}$. 5. $x^2-9\left\{\dfrac{1}{x+3}+\dfrac{2x-3}{x^2-3x+9}\right\}$.

6. $\dfrac{2x+2}{x^2+1}+\dfrac{5x-2}{(x^2+1)^2}-\dfrac{2}{x-1}$.

7. $\dfrac{5}{9(x-1)^2}-\dfrac{2}{3(x-1)}+\dfrac{6x+7}{9(x^2+x+1)}+\dfrac{x+3}{3(x^2+x+1)^2}$.

8. $\dfrac{x+1}{x^2+1}+\dfrac{2(x+1)}{(x^2+1)^2}+\dfrac{5x-2}{(x^2+1)^3}-\dfrac{1}{x-1}$.

9. $\dfrac{3x-6}{x^2+x+1}-\dfrac{2x-14}{(x^2+x+1)^2}+\dfrac{4x-10}{(x^2+x+1)^3}$.

10. $\dfrac{1}{x^2+4x+8}-\dfrac{8x}{(x^2+4x+8)^2}-\dfrac{55}{(x^2+4x+8)^3}$.

Page 164.

1. $\dfrac{(-1)^n \underline{|n}}{a^{n+2}}\sin^{n+1}\theta\sin(n+1)\theta$, where $x=a\cot\theta$.

2. $\dfrac{(-1)^n \underline{|n}}{a^{n+1}}\sin^{n+1}\theta\cos(n+1)\theta$.

3. $\dfrac{(-1)^n \underline{|n}}{b^{n+2}}\sin^{n+1}\theta\sin(n+1)\theta$, where $x-a=b\cot\theta$.

4. $\dfrac{(-1)^n \underline{|n}}{b^{n+2}}\sin^{n+1}\theta\{b\cos(n+1)\theta+a\sin(n+1)\theta\}$.

5. $\dfrac{(-1)^n \underline{|n}}{2^{n+1}}\sin^{n+1}\theta\{3\cos(n+1)\theta-\sin(n+1)\theta\}$, where $x+2=2\cot\theta$.

6. $\dfrac{(-1)^n \underline{|n}}{2^{n+5}}\sin^{n+1}\theta\{\sin(n+1)\theta-(n+1)\sin\theta\cos(n+2)\theta\}$.

7. $\dfrac{(-1)^n \underline{|n+1}}{2^{n+4}}\sin^{n+2}\theta\sin(n+2)\theta$.

8. $\mathrm{D}^n u=\dfrac{3}{8}\left(\mathrm{D}^n\dfrac{1}{x^2+1}+\mathrm{D}^{n+1}\dfrac{x}{x^2+1}\right)+\dfrac{1}{4}\mathrm{D}^{n+1}\dfrac{x}{(x^2+1)^2}$;

$\therefore\ \mathrm{D}^{-1}u=\dfrac{3}{8}\tan^{-1}x+\dfrac{3x}{8(x^2+1)}+\dfrac{x}{4(x^2+1)^2}$.

Page 167.

5. (1) $\log_e a+\dfrac{h}{a}-\dfrac{h^2}{2a^2}+\dfrac{h^3}{3a^3}-\ \ldots$.

(2) Put $a=1$, and $h=x$, in (1).

6. $\log\sin x+h\cot x-\frac{1}{2}h^2\operatorname{cosec}^2 x+\frac{1}{3}h^3\operatorname{cosec}^2 x\cot x-\ldots$.

Pages 185–187.

1. $1+\frac{1}{2}\theta^2+\frac{5}{24}\theta^4+\frac{61}{720}\theta^6+\ldots.$ **2.** $\frac{1}{2}\theta^2+\frac{1}{12}\theta^4+\frac{1}{45}\theta^6+\ldots.$

3. The coefficients of the even powers of x are all zero. Also $a_1=1.$ The equations giving the other terms are

$$a_3=a_1\,;\ \frac{a_5}{\underline{|3}}=a_3\left(1+\frac{1}{\underline{|2}}\right);\ \frac{a_7}{\underline{|5}}=a_5\left(\frac{1}{\underline{|3}}+\frac{1}{\underline{|4}}\right),\ \text{etc.},$$

whence $a_3=1$, $a_5=3^2$, $a_7=5^2.3^2$, etc.

4. $\theta=\sin\theta+\frac{\sin^3\theta}{\underline{|3}}+\frac{3^2\sin^5\theta}{\underline{|5}}+\ldots.$ **5.** $\theta=\tan\theta-\frac{1}{3}\tan^3\theta+\frac{1}{5}\tan^5\theta-\ldots.$

6. The function satisfies the differential equation $y_2=a^2y+xy_1+x^2y_2$, whence, finally,

$$y=1+ax+\frac{a^2x^2}{\underline{|2}}+\frac{(a^2+1)ax^3}{\underline{|3}}+\frac{(a^2+4)a^2x^4}{\underline{|4}}+\ldots\,;$$

the general relation being $a_{n+2}=(a^2+n^2)a_n$.

7. $bx+abx^2+\frac{1}{6}(3a^2b-b^3)x^3+\ldots.$

8. $1+ax+\frac{1}{2}(a^2-b^2)x^2+\frac{1}{6}(a^3-3ab^2)x^3+\ldots.$

Both (7) and (8) could have been obtained by equating the real and the imaginary parts in the identity $e^{ax}(\cos bx+i\sin bx)\equiv e^{(a+ib)x}$.

[The general terms of (7) and (8), respectively, may be written $\frac{(rx)^n}{\underline{|n}}\sin n\alpha$, and $\frac{(rx)^n}{\underline{|n}}\cos n\alpha$, where $r^2=a^2+b^2$, and $\tan\alpha=\frac{b}{a}.$]

9. $\frac{1}{2}(1-\cos 2x)=\frac{1}{2}\left\{\frac{(2x)^2}{\underline{|2}}-\frac{(2x)^4}{\underline{|4}}+\ldots\right\}.$

10. $\frac{1}{2}(\text{siu}\,3x-\sin x)=x-\frac{3^3-1}{2\underline{|3}}x^3+\frac{3^5-1}{2\underline{|5}}x^5-\ldots.$

11. $-\frac{1}{6}x^2-\frac{1}{180}x^4.$

14. (Express the chords in terms of the radius and the sines of half the angles which they subtend at the centre of the circle, and apply (13).)

15. $\theta=\frac{1}{180}(256\sin\theta-40\sin 2\theta+\sin 4\theta).$

16. $\theta+\frac{1}{6}\theta^3+\frac{1}{24}\theta^5+\ldots.$ **21.** See answer to No. 3 (above).

22. $$y = a_0\left(1 - \frac{x^3}{2.3} + \frac{x^6}{2.3.5.6} - \frac{x^9}{2.3.5.6.8.9} + \dots\right)$$
$$+ a_1\left(x - \frac{x^4}{3.4} + \frac{x^7}{3.4.6.7} - \frac{x^{10}}{3.4.6.7.9.10} + \dots\right).$$

23. $y = \mathrm{A}_0\left(1 - x + \frac{x^2}{2^2} - \frac{x^3}{2^2.3^2} + \dots\right).$

24. $y = \mathrm{A}_0\left(1 - \frac{x^2}{2^2} + \frac{x^4}{2^2.4^2} - \frac{x^6}{2^2.4^2.6^2} + \dots\right).$

25. $$y = \mathrm{A}_0\left(1 - \frac{(nx)^2}{\lfloor\underline{2}} + \frac{(nx)^4}{\lfloor\underline{4}} - \dots\right) + \frac{\mathrm{A}_1}{n}\left(nx - \frac{(nx)^3}{\lfloor\underline{3}} + \frac{(nx)^5}{\lfloor\underline{5}} - \dots\right)$$
$= \mathrm{A}\cos nx + \mathrm{B}\sin nx.$

Page 194.

3. $\frac{1}{2}$. 4. $\frac{16}{19}$. 5. $\frac{7}{5}$. 6. $\log_e a - \log_e b$.

7. (1) $\log_e x - \log_e a$; (2) $\log_e x$. 8. $\frac{5}{2}x^3$. 9. $-\frac{1}{2}x$.

10. 0. [Evaluate under the root.] 11. 2. 12. -1.

13. 2. 14. 2. 15. $2\sqrt{2}$. 16. $\frac{1}{2}x\sin x$. 17. $\frac{x\cos x - \sin x}{x\sec^2 x - \tan x}$.

Pages 197, 198.

1. *ratio* $=1$, *difference* $=0$. 2. ∞. 3. 0.

4. ∞. 5. $\frac{1}{2}n$. 6. $\frac{1}{2}$.

7. (1) e, 1; (2) e^{-1}, 1; (3) 1, 1; (4) 1, 1. 8. 1.

9. (1) $e^{\frac{1}{3}}$; (2) $e^{-\frac{1}{6}}$. 10. 1. 11. $\frac{6}{\pi}$. 12. 2.

Page 201.

1. a. 2. $b - a$. 3. $\frac{c}{23}$.

4. $\frac{8c-3}{9c+12}$. 5. (1) $c+2$; (2) $\frac{1}{2}c$. 6. $\frac{1}{2}(1-3c)$.

Page 204.

1. y^3 is comparable with x^6.
x^3y and xy^2 are comparable with x^5.

2. x^5 and x^3y^3 are comparable with z^{15}.
x^4y is comparable with z^{14}.
xy^5 is comparable with z^{13}.

Page 205.

1. (1) x^4, x^3y, $x^3+x^2y^2$, x^2y+xy^3, $x^2+xy^2+y^4$, $xy+y^3$, $x+y^2$, y.
(2) y^4, xy^3, x^2y^2, x^3y+y^3, x^4+xy^2, x^2y, x^3+y^2, xy, x^2, y, x.
(3) y^4, xy^3, x^2y^2, x^3y, x^4+y^3, xy^2, x^2y, x^3, y^2, xy, x^2, y, x.
(4) x^4, x^3y, x^3, x^2y^2, x^2y, x^2, xy^3, xy^2, xy, $x+y^4$, y^3, y^2, y.

2. Assuming that a *pair* of terms are to be of equal importance, the diagram will show that the only possible pairs are (1) *when x and y are great*; either x^3 and x^2y^2, or x^2y^2 and xy^3; (2) *when x and y are small*; either x^3 and x^2y, or x^2y and xy^3.

3. x^3 and y^2 are of greatest importance; then, in order, xy and x^2.

4. $2y-8$. 5. $3x-2y+8$.

Pages 207, 208.

1. $\dfrac{x^2+2xy-2y^2}{(x+y)^2}$; $-\dfrac{4x^2+2xy+y^2}{(x+y)^2}$.

2. $u\log_e a$; $ux \div a$. 3. pu/x; qu/y; $u\log_e x$; $u\log_e y$.

4. $2x-3y-5a$; $-3x+4y+6a$; $-5x+6y-2a$.

Pages 212, 213.

1. (i) $(2, -1)$; (ii) $(1, 1)$; (iii) $(-2, 3)$.

2. A parabola passing through the origin, whose axis is the line $5x-10y-4=0$, and whose vertex is at $\left(-\dfrac{28}{25}, -\dfrac{24}{25}\right)$.

3. The lines are $x-2y+a=0$, $2x-3y+a=0$, $x-y=0$. They all meet at (a, a). The *locus* consists of the lines $x-3y+2a=0$, and $x-y=0$.

4. The loci are the parabolas $x^2=ay$, and $y^2=ax$. The first cuts the curve in $(0, 0)$ and $(2^{\frac{1}{3}}a, 4^{\frac{1}{3}}a)$, and the second cuts it in $(0, 0)$ and $(4^{\frac{1}{3}}a, 2^{\frac{1}{3}}a)$.

Page 217.

1. (1) $xX+yY=a^2$; $xY-yX=0$.
(2) $yY=2a(X+x)$; $2a(Y-y)+y(X-x)=0$.
(3) $AxX+ByY=a^2$; $By(X-x)=Ax(Y-y)$.
(4) $yX+xY=2a^2$; $xX-yY=x^2-y^2$.

2. $(2x+A)X+(2y+B)Y+(Ax+By+2C)=0$.

3. Polar of (a, a) is $x^2+y^2=a^2$. The curve lies outside this circle, touching it at $(a, 0)$ and $(0, a)$.

4. $(y^2+a^2)h+(2xy-4ay)k=2ay^2-2a^2x$.

Polar of $(0, 0)$ is $y^2=ax$. It cuts the curve at $(0, 0)$ and $(a, \pm a)$. The tangents drawn from the origin are $x=0$ and $x=\pm y$.

Page 221.

1. $8y-6x$. **2.** $24xy^3$. **4.** $6u$; $9u$.

6. $\frac{d^2u}{dx^4}=\frac{d^2f}{dx^2}+2y_1\frac{d^2f}{dx\,dy}+y_1{}^2\frac{d^2f}{dy^2}+y_2\frac{df}{dy}$. **7.** $\frac{d^2y}{dt^2}=y_2$.

8. $x_1{}^2\frac{d^2f}{dx^2}+2x_1y_1\frac{d^2f}{dx\,dy}+y_1{}^2\frac{d^2f}{dy^2}$ In this example, $x_1=h$, and $y_1=k$.

9. $\left\{2\frac{df}{dx}\frac{df}{dy}\frac{d^2f}{dx\,dy}-\left(\frac{df}{dx}\right)^2\frac{d^2f}{dy^2}-\left(\frac{df}{dy}\right)^2\frac{d^2f}{dx^2}\right\}\div\left(\frac{df}{dy}\right)^3$.

Pages 229, 230.

1. *Minimum* at $(2, -1)$; viz. -4.

2. Critical point at (1, 1), but the function is neither a *maximum* nor a *minimum* there. (We shall call such a point a *double point.*)

3. *Minimum* at (2, 2); viz. -8. *Double point* at (0, 0).

4. *Maximum* at (3, 2); viz. 108. *Double point* at (0, 0).

5. *Maximum* at (0, 0); viz. *zero.*
Minimum at $(\sqrt{3}, -\sqrt{3})$, and at $(-\sqrt{3}, \sqrt{3})$; viz. -18.
Minimum at (1, 1) and $(-1, -1)$; viz. -2.

8. The series of ellipses are concentric, and all have the same axes, and evidently for a very small ellipse of the series the directions in which u increases most rapidly and least rapidly from its minimum value at the centre $(2, -1)$ are the directions of the minor and major axis respectively, and are given by the max. and min. values of $\frac{d^2u}{dr^2}$, since $\frac{du}{dr}$ is zero. These occur when $\tan 2\theta=\frac{2B}{A-C}$, which in this case is equal to -2.

9. u is constant in value $(=6)$ in the assigned directions, starting from the point (1, 1) where $\frac{du}{dx}$ and $\frac{du}{dy}$ are both zero. That is, the locus $u=6$ consists of two straight lines crossing at (1, 1) and lying in the assigned directions. The lines are $x-3y+2=0$ and $x-y=0$. [For all other values of u, the locus is a hyperbola having these lines as asymptotes.]

10. *Maximum* at $\left(\frac{2}{13}, \frac{3}{13}\right)$; viz. $\frac{13}{2}$.

Minimum at $(-2, -3)$; viz. $-\frac{1}{2}$.

12. (1) $h\frac{df}{dx}+k\frac{df}{dy}$ must be zero for all values of h and k; *i.e.* $\frac{df}{dx}$ and $\frac{df}{dy}$ must both vanish. (2) $\left(h\frac{d}{dx}+k\frac{d}{dy}\right)^2 f(x, y)$ must not be able to change sign, whatever the relation between h and k.

13. Let the successive terms in each expansion be denoted by $u, u_1, u_2, \ldots$. Then

(1) $u_2 = 4h^2 + 12hk + 10k^2$.
(2) $u_2 = h^2 - 4hk + 3k^2$.
(3) $u_2 = 3(h^2x + k^2y) - 6hk$.
$u_3 = h^3 + k^3$.
(4) $u_5 = -2h^4ky + h^3k^2(6 - 4x - 3y) - 3h^2k^3x$.
$u_6 = -h^4k^2 - h^3k^3$.
(5) $u_2 = 6(h^2x^2 + k^2y^2) - 4h^2 + 4hk - 4k^2$.
$u_3 = 4(h^3x + k^3y)$.
$u_4 = h^4 + k^4$.

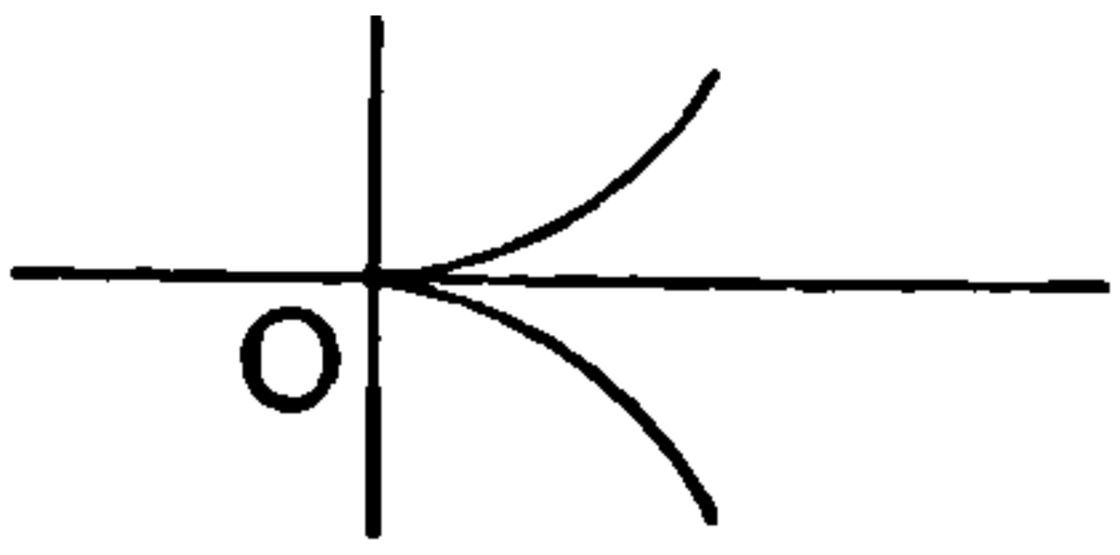
14. (1) $4x^2 + 12xy + 10y^2 - 4$ when $h = 2$, $k = -1$.
(2) $x^2 - 4xy + 3y^2 + 6$ when $h = 1$, $k = 1$.

15. When $u = -8$ the equation of the locus can be written in the form $(x + y + 2)(x^2 - xy + y^2 - 2x - 2y + 4) = 0$. [See also Ex. 6, p. 217.]

Page 235.

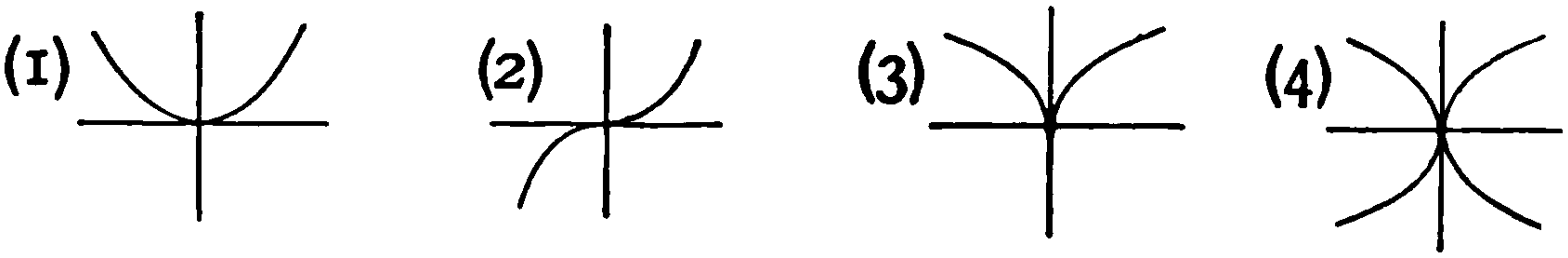
1. (1) $3x - 2y = 0$. (3) $y = 0$.
(4) $x + y = 0$; $x - y = 0$. (5) $x + y = 0$.

2.



Such a point is called a cusp, and so is 3 (3) below.

3.



Page 244.

1. (2) The curve cuts the sloping asymptotes in the points $(5, \pm\frac{11}{2})$. It does not cut the vertical asymptote.
(3) Cuts the asymptote at $(\frac{4}{3}, \frac{2}{3})$.
(4) In no finite point.
(5) In no finite point.

4. If the compound equation of the asymptotes is $u = 0$, the equation of the curve must be of the form $u = Ax + By + C$. Therefore the points of intersection lie on the line $Ax + By + C = 0$. (See Examples 3 and 5.)

Page 246.

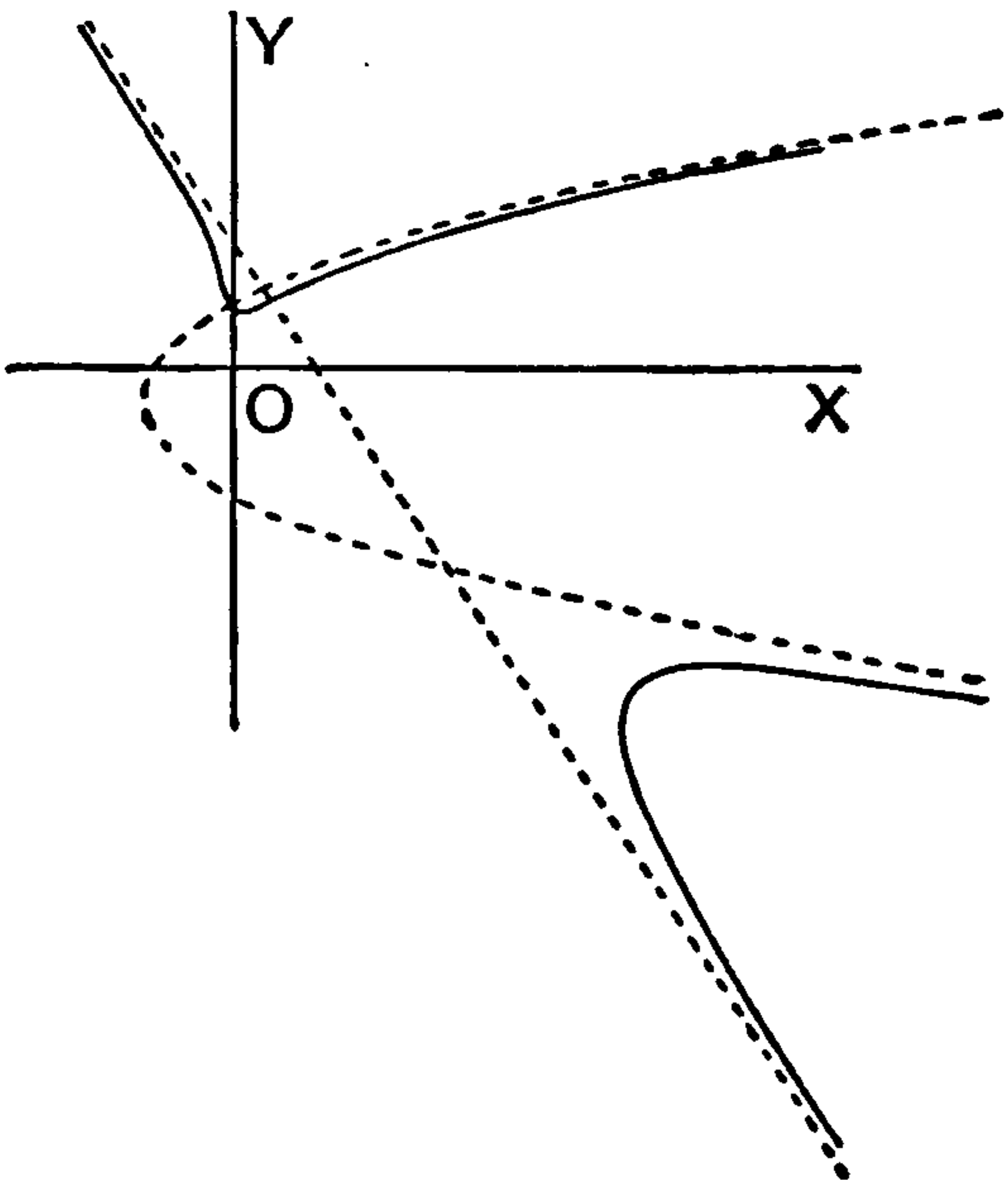
1. Cuts linear asymptote at $(\frac{4}{3}, 2)$; curvilinear at $(0, 2)$.

2. Zero gradient at $(8 \mp \frac{16}{3}\sqrt{2}, -4 \pm 4\sqrt{2})$, *i.e.* at $(0·46, 1·66)$, and $(15·54, -9·66)$. Infinite gradient at $(12·6, -11·3)$.

The tangent at $(0, 2)$ is parallel to the asymptote.

4. The equation can be written $y^2(3x+2y-4)=9x^2$, so that $3x+2y-4$ cannot be negative, unless x and y are both zero.

6. The curve has two branches, on one of which are the points $(-1, 5)$, $(0, 2)$, $(\frac{4}{3}, 2)$, and $(10, 5)$, and on the other branch are the points $(13, -13)$, $(13\frac{1}{3}, -10)$, $(20, -10)$. These data, together with the asymptotes, and the points given in Ex. 2 will be sufficient to give a very good idea of the curve, which is shown below. The student should draw it to a larger scale, from his own calculations.



Page 251.

1. $\tan^{-1}b/a$; $\sqrt{(a^2+b^2)}$.

4. Points on the curve can easily be found by drawing the circle $r=a\cos\theta$, and adding the length a to each radius vector. The curve has a cusp at the origin.

Page 254.

1. $r^2=ap$. 2. $p^2=ar$.

Page 256.

1. $p^2=ar$. $\rho=2pr\div a=2r^{\frac{3}{2}}a^{-\frac{1}{2}}$. (1) $2a$; (2) $4a\sqrt{2}$.

2. $\frac{1}{2}a$ (the curve is a circle).

3. $2ap^2=r^3$. $\rho=4ap\div 3r=\frac{2}{3}\sqrt{(2ar)}$. 4. $r^3=a^2p$. $\rho=a^2\div 3r$.

5. $r^{n+1}=a^np$. $\rho=a^n\div(n+1)r^{n-1}$.
For the cardioid, $n=\frac{1}{2}$; for the parabola, $n=-\frac{1}{2}$.

ANSWERS TO THE MISCELLANEOUS EXAMPLES.

Pages 257–263.

2. (1) $\dfrac{6(1-2x^2)}{(2x^2-3x+1)^2}$; (2) $2\sqrt{(x^2+1)}$.

3. The values are 6641 and 6690, approx. **4.** 1·78.

5. $\dfrac{1}{\pi}\tan^{-1}\dfrac{b}{a}$; $b=\tan 72^\circ=3{\cdot}078$.

6. $$\text{Angular velocity}=\frac{(au+bv)-(av+bu)\cos \text{POQ}}{a^2+b^2-2ab\cos \text{POQ}}.$$

[One way of obtaining this is to differentiate the gradient, $\tan\phi$, of PQ, taking $(a\cos\alpha,\ a\sin\alpha)$ as the coordinates of P, where $a\alpha=ut$, and similarly for Q. After some straightforward reduction, the value of $d\phi/dt$, given above, is found.] For reversal, it must be possible for $d\phi/dt$ to become zero, which requires

$$av+bu>au+bv,\ i.e.\ (a-b)(v-u)>0,$$

i.e. the linear velocity in the smaller circle must be greater than the other.

7. 140 $(ft)^3$, nearly. $(400\pi\div 9)$.

8. He must leave the road 346 *ft*, $(200\sqrt{3})$, before reaching N, where PN is the perpendicular from P to the road.

9. $7\times 3^{\frac{1}{3}}\div(3^{\frac{1}{3}}+4^{\frac{1}{3}})=3\tfrac{1}{3}$ feet; $14\sqrt{3}-21=3\tfrac{1}{4}$ feet.

10. ρ is infinite at $(2,\ \frac{8}{5})$. If the origin is transferred there, $y=\frac{1}{10}(12x-x^3)$, which is unaltered if $(-x,\ -y)$ is substituted for $(x,\ y)$.

11. $\dfrac{dy}{dx}=\dfrac{a'}{a}-\dfrac{2c}{a^2}(x-b)$.

12. The given lines are $\dfrac{du}{dx}=0$, and $\dfrac{du}{dy}=0$, corresponding to max. and min. values of y and x respectively.

13. Transform to polar coordinates: $\tan 2\theta=-1{\cdot}5$; r is max. when $\theta=-28^\circ\ 9'$; min. when $\theta=61^\circ\ 51'$.

14. $\cos x\div 3437$; $\cos 69^\circ\ 54'={\cdot}3437$, and $\cos 88^\circ\ 2'={\cdot}03432$; $\therefore$ a 5 figure table would suffice up to over 88°.

15. $\text{A}=3a^4-7a^3+12$; $\text{B}=12a^3-21a^2$; $\text{C}=18a^2-21a$; $\text{D}=12a-7$; $\text{E}=3$.

16. $m(x-h)\sqrt{k}-(y-k)\sqrt{h}=0$; and $(x-h)\sqrt{h}+m(y-k)\sqrt{k}=0$.

17. $\dfrac{\sqrt{(a^2+4x^2)}}{x}-\dfrac{a^2}{2x^2}\log_e\dfrac{2x+\sqrt{(a^2+4x^2)}}{a}$, $=20\cdot09975-19\cdot9666=\cdot1331$ when $a=100$, $x=5$. The reciprocal of this is 7·513.

The limit for s is $2a$, when $x=0$.

18. $\dfrac{d^2y}{dx^2}=0$. The difference of ordinates, $(y_2-y_1)=(x-1)^3$, showing that y, $\dfrac{dy}{dx}$, and $\dfrac{d^2y}{dx^2}$ are the same for both curves, and that y_2-y_1 changes sign as x increases through the critical point. The radius of curvature $=17\sqrt{17}\div 8$.

19. $\rho=2a$.

The curve can be written $\pm\sqrt{x}\pm\sqrt{y}=\sqrt{a}$, therefore we may put

$$x=at^2,\quad y=a(1-t)^2,$$

whence the tangent is $ty+(1-t)x=at(1-t)$, and the sum of its intercepts $=a$.

20. $y=\sec x-\tan x$, whence $y_2+yy_1=0$, whence by repeated differentiations, the successive coefficients can be obtained. The expansion is

$$1-x+\frac{x^2}{\lfloor 2}-\frac{2x^3}{\lfloor 3}+\frac{5x^4}{\lfloor 4}-\frac{16x^5}{\lfloor 5}+\ldots.$$

[See also separate expansions of $\sec x$ and $\tan x$.]

21. (1) $\rho=\dfrac{\sqrt{2}}{a+p}$; (2) there are two branches, each having $\rho=2\sqrt{2}$.

22. The roots are 0·174, 0·766, and −0·940.

23. 400 lbs. per sq. foot.

24. $x^2+1=400\sqrt{(x-10)}$. The real roots must exceed 10 and be less than 52. They are 10·07 and 50·42.

25. Perpendicular to axis at $\left(-\frac{1}{4},\ \frac{1}{2}\right)$ and $\left(\frac{1}{4},\ \frac{3}{2}\right)$. Point of inflexion at $(0,\ 1)$.

26. 102 sq. ft.

28. $x^2y_{n+2}+(2n+1)xy_{n+1}+(n^2+1)y_n=0$.

31. If the normal distance between two adjacent contours is represented by dn, $\left(\dfrac{dz}{dn}\right)=\sqrt{\left\{\left(\dfrac{dz}{dx}\right)^2+\left(\dfrac{dz}{dy}\right)^2\right\}}$; $\therefore$ the expression gives the gradient of the line of steepest slope at any point of the surface. Its greatest values are where the planes $y=\dfrac{\pi}{2}$ and $x=\dfrac{\pi}{2}$ cut the surface, and its least where it is cut by the plane $x=y$.

32. $x=\frac{1}{3}$.

33. $\frac{1}{2}(x-1)(x+2)^2$ Max. when $x=-2$; min. when $x=0$.

34. *Hint.* Differentiate $x=f(u, v)$, $y=\phi(u, v)$ partially with regard to x. Multiply by $\frac{du}{dx}, \frac{du}{dy}$ respectively, and make use of the orthogonal condition.

35. y is max. at $(\frac{1}{2}, \frac{1}{2}e^{-1})$; pt. of inflexion at $(1, e^{-2})$.

36. The curve cuts $y=0$ at right angles where $x=2a\div\pi$, and $y=0$ is an axis of symmetry. The axis of y is cut at the gradient $-2\div\pi$, where $y=a$. x is negative between $y=a$, and $y=2a$, the latter being an asymptote.

37. *Hint.* Interest + cost of pumping $=\mathrm{A}d^2+\mathrm{B}d\,.\,v^{\frac{3}{2}}$, where A, B are constants. Also quantity of water is constant, *i.e.* $d^2.v=\mathrm{C}$.

38. Length $=a$. [Gradient $=\sqrt{a^2-x^2}\div x$.]

39. Thickness of slice cut off $=\frac{1}{3}$ radius.

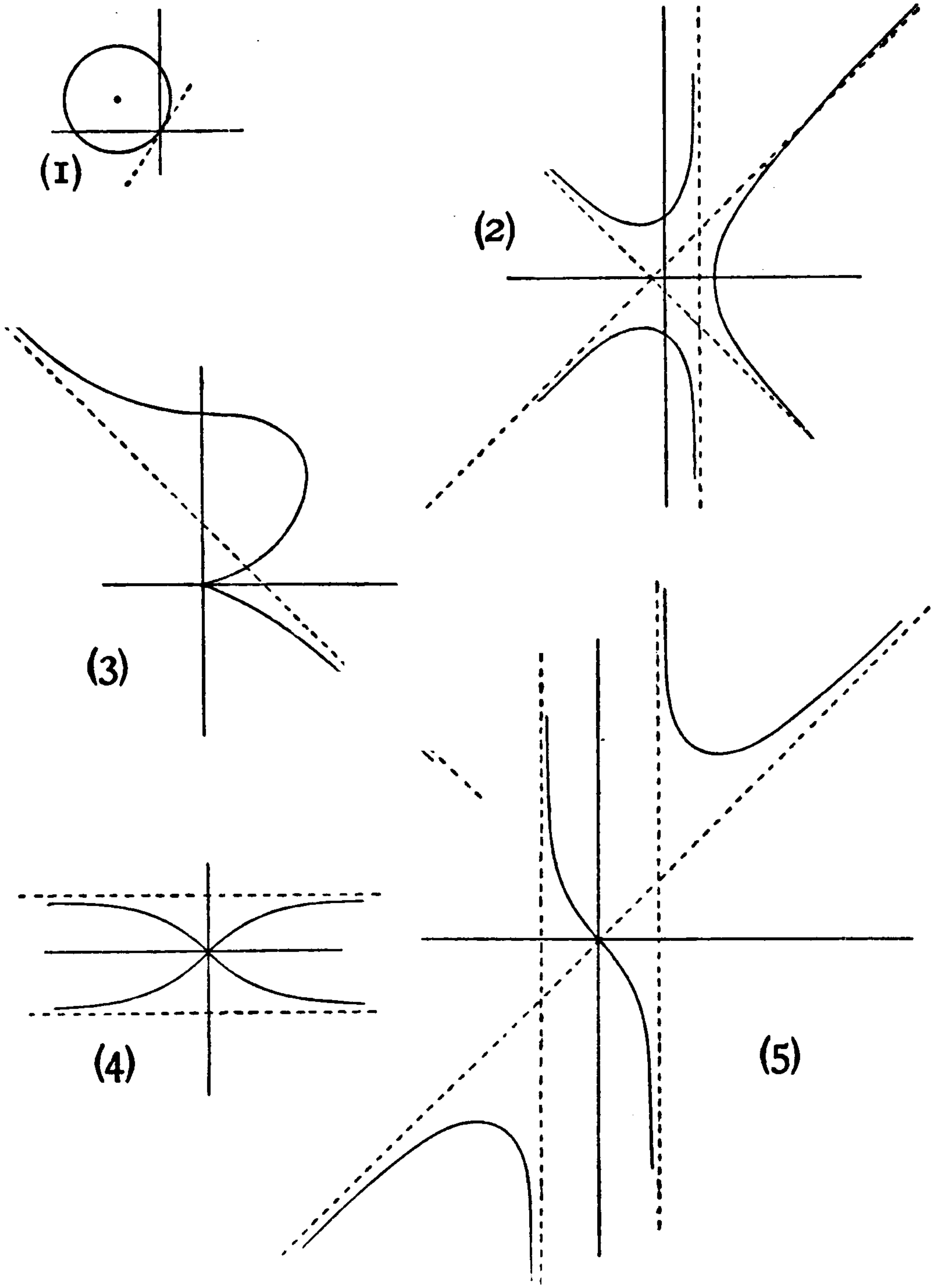
40. *Hint.* Obtained by taking the tangents at a, and $a+\delta a$, and solving simultaneously.

41. [One method is to put $x=\sin 3\theta$, $y=\sin\theta$.]

48. $x=a-\frac{F(a)-f(a)}{F'(a)-f'(a)}$.

50. (2) *Hint.* Expand the right hand operator, and make use of the fact that $\left(\frac{d}{dx}+\frac{d}{dy}+\frac{d}{dz}\right)^2 u=0$, which follows from (1).

52. *Hint.* Take logarithms before performing the partial differentiations.



Graphs of Curves given in Art. 198.

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